1. Let $X_1, \ldots, X_n$ be i.i.d. random variables, so that X_1 has probability density function $f_\theta \colon \mathbb{R} \to [0, \infty)$, where $\theta > 0$ is an unknown parameter and

$$f_\theta(x) := \begin{cases} \frac{2x}{\theta^2}, & 0 \le x \le \theta, \\ 0, & \text{else.} \end{cases}$$

   (a) Find any method of moments estimator $\widehat{\theta}_n$ of θ. Is $\widehat{\theta}_n$ unbiased?

   (b) Show that $\widehat{\theta}_n$ converges in probability as $n \to \infty$.

   (c) Show that $\widehat{\theta}_n$ converges in distribution as $n \to \infty$, and identify the limiting distribution.

   (d) Prove or disprove the following statement: let $W_1, W_2, \ldots$ be real random variables that converge in distribution to W. Let $Z_1, Z_2, \ldots$ be real random variables that converge in distribution to Z. Then $W_1 + Z_1, W_2 + Z_2, \ldots$ converges in distribution to $W + Z$.

2. Assume that $Y_1, \ldots, Y_n$ are independent and generated from a linear model $Y_j = \alpha + \beta x_j + \varepsilon_j$, where $\alpha, \beta \in \mathbb{R}$ are unknown, x_j's are not all equal, and $\varepsilon_j,\ j = 1., \ldots, n$ are i.i.d. $N(0,1)$ random variables.

   (a) Write down the likelihood function and find a complete, sufficient statistic for (α, β). Justify both sufficiency and completeness.

   (b) Find the maximum likelihood estimator (MLE) of the pair (α, β).

   (c) Show that the MLE is unbiased.

   (d) Show that the MLE has the smallest variance among all unbiased estimators.