Geometry and Topology Graduate Exam
Fall 2022

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Consider the vector fields $X = e^x \partial_x$ and $Y = \partial_y$ on $\mathbb{R}^2$. Find all vector fields Z on $\mathbb{R}^2$ such that $[X, Z] = [Y, Z] = 0$.

Problem 2. Let X be a path-connected topological space, and let $x \in X$. Show that $\pi_1(X, x)$ is trivial if and only if, for any $x_1, x_2 \in X$, any two paths $\gamma, \delta\colon [0,1] \to X$ from x_1 to x_2 are homotopic (through paths from x_1 to x_2).

Problem 3. Let $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ be the 2-torus, and let $\alpha, \beta, \gamma \in \Omega^1(\mathbb{T}^2)$ be closed 1-forms on $\mathbb{T}^2$. Show that there exist real numbers $a, b, c \in \mathbb{R}$ such that $a\alpha + b\beta + c\gamma$ is exact.

Problem 4. Let ω be a 1-form on the sphere $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Show that if ω is invariant under rotations, i.e. $\phi^*\omega = \omega$ for all $\phi \in SO(3)$, then $\omega = 0$.

Problem 5. Show that if M and N are compact, connected smooth manifolds, then every submersion $f\colon M \to N$ is surjective.

Problem 6. Let X be the complement of a point in $S^1 \times S^1 \times S^1$. Calculate the fundamental group and homology groups of X.

Problem 7. Show that every continuous map from $\mathbb{RP}^2 \times \mathbb{RP}^2$ to $S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.