REAL ANALYSIS GRADUATE EXAM

Fall 2022

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable so that $0 < A = \int_E f(x)\,dx < \infty$. Show that for every $t \in (0,1)$ there exists a measurable set $E_t \subset E$ so that $\int_{E_t} f(x)\,dx = tA$.

2. Assume that $f \in \mathscr{L}^1(\mathbb{R})$ and let $F(t) = \int f(x)e^{itx}\,dx$. Prove that $F : \mathbb{R} \to \mathbb{R}$ is continuous and that $\lim_{t\to\infty} F(t) = \lim_{t\to-\infty} F(t) = 0$.

3. Compute the following limit and justify your calculations

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.$$

4. For $f \in \mathscr{L}^1(\mathbb{R})$ consider the maximal function

$$\mathcal{M}f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\,dt.$$

Prove that there exists a constant $A > 0$ so that for any $\alpha > 0$

$$m(\mathcal{M}f > \alpha) \le \frac{A}{\alpha} |f|_{\mathscr{L}^1}$$

where m is the Lebesgue measure on $\mathbb{R}$.