Algebra Exam August 2022

Show your work. Be as clear as possible. Do all problems.

1. Let G be the quaternion group of order 8.
   (a) Determine the algebra structure of $\mathbb{R}[G]$.
   (b) Determine the algebra structure of $\mathbb{C}[G]$.

2. Let R be a commutative ring with 1. Let $r_1, \ldots, r_n \in R$ which generate R as an ideal. Let $f : R^n \to R$ be defined by $f(a_1, \ldots, a_n) = \sum_i a_i r_i$. Show that the kernel of f is a projective module.

3. Let G be a finite group with a cyclic Sylow 2-subgroup S.
   (a) Show that $N_G(S) = C_G(S)$.
   (b) Show that if $S \neq 1$, then G contains a normal subgroup of index 2 (hint: suppose that $n = [G : S]$, consider an appropriate homomorphism from G to S_n).
   (c) Show that G has a normal subgroup N of odd order such that $G = NS$.

4. Let R be a principal ideal domian and $p \in R$ a prime element. Suppose that V is a finitely generated R-module with $p^a V = 0$ and suppose $v \in V$ with the annihilator of v in R the ideal $p^a R$. Prove that $V = Ra \oplus W$ for some submodule W of V.

5. Let $f(x) = x^7 - 3 \in \mathbb{Q}[x]$.
   (a) Show that f is irreducible in $\mathbb{Q}[x]$.
   (b) Let K be the splitting field of f over $\mathbb{Q}$. What is the Galois group of $K/\mathbb{Q}$.
   (c) How many subfields L of K are there so such that $[K : L] = 7$.

6. Let M be a maximal ideal of $\mathbb{Q}[x_1, \ldots, x_t]$.
   (a) Show for each i, there exists a nonzero polynomial f_i with coefficients in $\mathbb{Q}$ such that $f_i(x_i) \in M$.
   (b) Show that there are only finitely many maximal ideals of $\mathbb{C}[x_1, \ldots, x_t]$ which contain M.