MATH 507a **QUALIFYING EXAM** **Fall 2022**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Let $\{x_j, j \geq 1\} \subset \mathbb{R}$, let $a_j > 0$ with $\sum_{j\geq 1} a_j = 1$, and let F be the distribution function with a jump of size a_j at each x_j. Suppose F_n has the same first n jumps, of size a_j at x_j for each $j \leq n$. (F_n may also have other jumps or may have a continuous part.) Show that $F_n \to F$ weakly.

(2) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}, \mathcal{H}$ be sub–σ–algebras of $\mathcal{F}$ which are independent, that is, $P(A \cap B) = P(A)P(B)$ for all $A \in \mathcal{G}, B \in \mathcal{H}$. Let X be a random variable with $E|X| < \infty$. Find

$$E\Big(E(X \mid \mathcal{H}) \Bigm| \mathcal{G}\Big).$$

HINT: What is the integral of this function over a set in $\mathcal{G}$?

(3) Let $X_0, X_1, \ldots$ be iid with finite mean, not identically 0. Show that with probability 1, the infinite series

$$\sum_{n=0}^{\infty} X_n z^n$$

converges for all $z \in \mathbb{C}$ with $|z| < 1$, and diverges for all $z \in \mathbb{C}$ with $|z| \geq 1$.

HINT: Considering $z \in \mathbb{C}$ affects things very little—if we considered only $z \in \mathbb{R}$ the proof would be nearly all the same.