MATH 505a **QUALIFYING EXAM** **Fall 2022**

Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1)(a) A standard deck of 52 cards has 13 cards of each suit (spades, hearts, diamonds, clubs.) A bridge hand consists of 13 randomly chosen cards from the deck (chosen without replacement.) The bridge hand is said to be *void* in a suit if it contains no cards of that suit. Find the probability that a bridge hand is void in at least one suit.

(b) Let A be an 10-element set and let B be a 4-element set, and let Φ be a random map from A to B. Show that

$$P(\Phi \text{ is not a surjection}) \leq 4 \cdot \left(\frac{3}{4}\right)^{10}.$$

Here a "random map" means one chosen uniformly among all possible maps.

(c) Find an exact formula for the probability in (b).

Your answers to (a) and (c) may involve factorials and/or quantities $\binom{n}{k}$; you need not simplify these any further.

(2) Let $n \geq 5$ and let $X_1, \ldots, X_n$ be iid, each taking integer values 0 through 9 with probability 1/10 each. Let N be the number of indices i for which $X_i = X_{i+1} = X_{i+2}$. Find the mean and variance of N.

Note there are no variables $X_{n+1}, X_{n+2}, \ldots$ defined, just X_1 through X_n. Also, N counts strings of 3 identical digits; if you get a string of more than 3 identical digits, there will be overlapping strings of 3 digits. So for example ..744442.., contributes 2 to the value of N, because 444 appears twice.

(3) Let $U_1, \ldots, U_n$ be iid uniform in [0,1], and let X_n be the second smallest of the values $U_1, \ldots, U_n$.

(a) Find $P(X_n > t)$ for $t \in [0, 1]$.

(b) Find c_n so $c_n - \log X_n$ converges in distribution, and find the distribution function of the limit. HINT: Given $a \in \mathbb{R}$, what is $\lim_n (1 + \frac{a}{n})^n$?