Numerical Analysis Preliminary Examination Fall 2022

August 25, 2022

Problem 1.
(a) Show that if $A \in \mathbb{C}^{n\times n}$ is strictly diagonally dominant -i.e., $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$, $i = 1, 2, ..., n$ then A is nonsingular.

(b) Provide a nontrivial example, (that is a matrix with no zero rows) of a diagonally dominant matrix that is singular.

(c) Suppose that $A \in \mathbb{C}^{n\times n}$ is strictly diagonally dominant with $a_{ii} \in \mathbb{R}$ and $a_{ii} > 0$, $i = 1, 2, ..., n$. Show that every eigenvalue of A has positive real part.

(d) Let $A \in \mathbb{C}^{n\times n}$, and suppose that there is a $k \in \{1, 2, \ldots, n\}$ for which $|a_{kk}| \geq \sum_{j\neq k} |a_{kj}|$ and that $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for $i \neq k$. Show that A is nonsingular. (Hint: Apply Gershgorin's Theorem to the matrix $B = D^{-1}AD$, where D is an appropriately chosen diagonal matrix).

Problem 2.
Show that if $A \in \mathbb{C}^{n\times n}$ is diagonalizable with $A = E\Lambda E^{-1}$, $B \in \mathbb{C}^{n\times n}$ and $\hat{\lambda}$ is an eigenvalue of $A + B$, then there is an eigenvalue λ of A such that $|\lambda - \hat{\lambda}| \leq \kappa(E)\|B\|_\infty$, where $\kappa(E)$ denotes the condition number of the matrix E with respect to the infinity matrix norm, $\|\cdot\|_\infty$, on $\mathbb{C}^{n\times n}$.

Problem 3.
Let x be the least squares solution to the problem $Ax = b$, where A is a $m \times n$ real matrix of rank n.

(a) Derive the normal equations and write the solution x as $x = A^\dagger b$ where $A^\dagger$is the pseudo-inverse of A. That is find a compact expression for $A^\dagger$ in terms of A.

(b) Let $\tilde{b} = b + \delta b$ be a perturbation of the vector b. The matrix A remains unchanged. Let $\tilde{x}$ be the least squares solution to $A\tilde{x} = \tilde{b}$. Show that if $b_R \neq 0$, then $\frac{\|\tilde{x}-x\|}{\|x\|} \leq \text{cond}(A)\frac{\|\delta b_R\|}{\|b_R\|}$, where $\text{cond}(A) = \|A\|\|A^\dagger\|$, b_R and δb_R are, respectively, the projections of the vectors b and δb onto $R(A)$.

In the following questions, give your answers in terms of fractions and square roots and whenever A, b or δb are mentioned, use:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \delta b = 10^{-3} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

(c) Use Gauss elimination to find the least squares solution of $Ax = b$.

(d) Find and use the Cholesky decomposition to find the least squares solution of $Ax = b$.

(e) Use part (b) to get an upper bound on $\frac{\|\tilde{x}-x\|}{\|x\|}$. Use any induced norm you like.

(f) Find and use a full or reduced QR decomposition of A to find the least squares solution of $Ax = b$. Use your favorite method to find this QR decomposition.

(g) Find and use the SVD of A to find the least squares solution of $Ax = b$. Use your favorite method to find this SVD.