ODE EXAM - Fall 2021

The exam has **four** problems. Each problem is worth 10 points. Do all four problems.

1. Show that, for every $a \in [-2, 1]$, the solution $y = y(t)$ of the equation

$$y' = e^y - 1 - 3y,$$

   with initial condition $y(0) = a$, is defined for all $t > 0$ and satisfies $|y(t)| \le |a|$.

2. Consider the linear system

$$X'(t) = \big(A + \varepsilon B(t)\big)X(t),$$

   where the matrix $A \in \mathbb{R}^{n\times n}$ is constant and symmetric, ε is a real number, and the entries of the matrix $B = B(t) \in \mathbb{R}^{n\times n}$ are bounded continuous functions of t for all $t \ge 0$. Denote by $|X(t)|$ the Euclidean norm of the vector $X(t)$.

   Show that if all eigenvalues of A are strictly negative and $|\varepsilon|$ is sufficiently small, then all solutions of the system satisfy

$$\lim_{t\to+\infty} |X(t)| = 0.$$

3. For the nonlinear system

$$\begin{cases} x' = xy^2 + x^3 \\ y' = x^2y + y^3, \end{cases}$$

   identify the critical points and determine their stability.

4. Let $f = f(t)$ be a continuous periodic function with period 1. Consider the second-order equation

$$y''(t) + f(t)y(t) = 0,$$

   and let $u = u(t)$ and $v = v(t)$ be two solutions of the equation with initial conditions $u(0) = v'(0) = 1$, $u'(0) = v(0) = 0$. Show that if $u(1) + v'(1) > 2$, then the equation has no non-trivial solutions that remain bounded for all $t \in (-\infty, +\infty)$.