PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Fall 2021

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $U \subset \mathbb{R}^n$ be an open set and let $u\colon U \to \mathbb{R}$ be a harmonic function in U such that $u(x_0) + u(y_0) = M$ for some $M \in \mathbb{R}$, and $x_0, y_0 \in U$. Show that there exist infinitely many pairs $(x, y) \in U \times U$ such that $u(x) + u(y) = M$.

2. Let $U \subset \mathbb{R}^n$ be open and bounded and let $U_T := U \times (0, T)$. Let $u : \overline{U_T} \to \mathbb{R}$ be continuous, C^1 with respect to time variable and C^2 with respect to the spatial variables, in U_T. Suppose that u satisfies

$$u_t - \Delta u \leq 0 \qquad \text{in } U_T.$$

Show that

$$\max_{\overline{U_T}} u = \max_{\partial U \times [0,T] \cup U \times \{0\}} u.$$

3. Let u be a classical solution of the problem

$$\begin{cases} u_{tt} = u_{xx} & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are smooth functions and have compact supports. Show that there exists $t_0 > 0$ such that

$$\int_{-\infty}^{+\infty} u_t^2(x,t)dx = \int_{-\infty}^{+\infty} u_x^2(x,t)dx$$

for all $t > t_0$.