1. Let $\boldsymbol{X} \sim \mathcal{N}_n(\mu, \Sigma)$, that is, $\boldsymbol{X}$ has the multivariate normal distribution in $\mathbb{R}^n$ with mean μ and covariance matrix Σ, and for all $\alpha \in (0,1)$ defined z_α by $P(Z \le z_\alpha) = \alpha$ where $Z \sim \mathcal{N}(0,1)$.

   (a) Let Σ be positive definite, and for fixed μ_1 and μ_2, distinct vectors in $\mathbb{R}^n$, express the Neyman-Pearson test with type I error $\alpha \in (0,1)$ of $H_0 : \mu = \mu_0$ versus $H_1 : \mu = \mu_1$ in terms of a test statistic $T(\boldsymbol{X})$ that has the $\mathcal{N}(0,1)$ distribution under H_0.

   (b) Find the power function β of the test in (a) and determine what it specializes to in the case where the components of $\boldsymbol{X}$ are independent and identically distributed univariate normal distributions.

   (c) Let Σ be a non-zero, non-negative definite covariance matrix that is not positive definite. Show that for some values of μ_0, μ_1 that there exists tests for the hypotheses in part (a) that have type I error $\alpha = 0$ and power $\beta = 1$. Determine a set of necessary and sufficient conditions on μ_0, μ_1 for that to be the case, prove that your conditions are as claimed, and give the form of these $\alpha = 0, \beta = 1$ tests.

2. Let $X_1, \ldots, X_n$ be i.i.d. with Bernoulli(p) distribution for some $p \in (0,1)$, meaning that $P(X_1 = 1) = p,\ P(X_1 = 0) = 1 - p$.

   (a) Show that the maximum likelihood estimator of p is $\bar{X}_n = \frac{1}{n}\sum_{j=1}^n X_j$.

   (b) For some $p_0 \in (0,1)$, we would like to test the null hypothesis $H_0 : p = p_0$ against the alternative $H_a : p \neq p_0$ using the likelihood ratio test. The log-likelihood ratio test statistic is defined as $\Lambda_n = \log\left(\frac{\sup_{p\in[0,1]} L_n(p)}{L_n(p_0)}\right)$ where $L_n(p)$ is the likelihood function, that is, the joint density of the observations, considered as a function of p. Show that the explicit expression for this test statistic is

   $$\Lambda_n = n\left(\bar{X}_n \log\left(\frac{\bar{X}_n}{p_0}\right) + (1 - \bar{X}_n)\log\left(\frac{1-\bar{X}_n}{1-p_0}\right)\right).$$

   (c) Prove that

   $$2\Lambda_n = \frac{n\left(\bar{X}_n - p_0\right)^2}{p_0(1-p_0)} + o_P(1)$$

   where $o_P(1)$ is a term that converges to 0 in probability as $n \to \infty$. Deduce from this expression the approximate distribution of $2\Lambda_n$ for large n.

   For this task, you may use the following facts without proving them: $\bar{X}_n \log\left(\frac{\bar{X}_n}{p_0}\right) = (\bar{X}_n - p_0 + p_0)\log\left(1 + \frac{\bar{X}_n - p_0}{p_0}\right)$, $\log(1+x) = x - x^2/2 + o(x^2)$ when $x \to 0$ and $n \cdot o\left((\bar{X}_n - p_0)^2\right)$ converges to 0 in probability as $n \to \infty$ (here, $o(\cdot)$ stands for "small-O").