Geometry and Topology Graduate Exam
Fall 2021

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Find all of the 2-sheeted covering spaces (connected or disconnected) of $S^1 \times S^1$, up to isomorphism of covering spaces without basepoints.

Problem 2. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be the function defined by

$$f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

(a) Find a real number r such that $f^{-1}(r)$ is a smooth manifold and prove it.
(b) Find a real number r such that $f^{-1}(r)$ is not a smooth manifold and prove it.

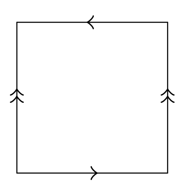
Problem 3. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ be the inclusion. Compute the integral over S^2 of the restriction

$$\int_{S^2} \omega = \int_{S^2} i^* \omega$$

of the 2-form on $\mathbb{R}^3$ given by $\omega = 2x^2 dx \wedge dz - xdy \wedge dz + 3ydx \wedge dz$.

Problem 4. Let $\mathcal{D}$ be the distribution on $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 | y > 0\}$ given by the kernel of the 1-fom $\alpha = dz - \log(y)dx$. Is $\mathcal{D}$ integrable? Provide justification.

Problem 5. Let K be the Klein bottle (the closed square with boundary identifications as pictured below).



(a) Let $p \in K$ be the image of some point in the interior of the closed square (under the identifications above). Say whether the following assertion is true or false (and give justification): $K \backslash \{p\}$ is homotopy equivalent to $S^1 \vee S^1$.
(b) Show that K is homeomorphic to the disjoint union of two Möbius bands with the boundary circles identified.
(c) Use part (b) (whether or not you solved it) to compute $\pi_1(K)$ via van Kampen's theorem and the integral singular homology $H_*(K; \mathbb{Z})$ via the Mayer-Vietoris long-exact sequence.

Problem 6. A *space-filling curve* is a continuous surjective map $f : \mathbb{R} \to \mathbb{R}^2$ (it is a classical fact that such curves exist).

(a) Prove that if f is any such space-filling curve, then f cannot be smooth. Equivalently, prove that if $f : \mathbb{R} \to \mathbb{R}^2$ is any smooth map, then f cannot be surjective.
(b) Prove that if f is any space-filling curve, then f cannot be a homeomorphism.

Problem 7. Let X be the space given by taking the circle S^1 and attaching two 2-cells to S^1 along degree 9 and 12 attaching maps respectively, and then identifying a point in the interior of the first 2-cell with a point in the interior of the second 2-cell. Compute the integral homology $H_*(X;\mathbb{Z})$ in every degree.