REAL ANALYSIS GRADUATE EXAM

Fall 2021

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f be integrable on $\mathbb{R}^d$ with respect to the Lebesgue measure m. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that A Lebesgue measurable and $m(A) < \delta$ imply $\int_A |f(x)|dx < \epsilon$.

2. Assume that f is integrable on $\mathbb{R}$ with respect to the Lebesgue measure. Prove that

$$\lim_{h\to 0} \int |f(x+h) - f(x)|\, dx = 0.$$

3. Assume that f and $\{f_n\}_{n=1}^{\infty}$ are Lebesgue measurable on $\mathbb{R}$ and that we have

$$\int |f(x) - f_n(x)|\, dx \leq \frac{C}{n^2}, \qquad n \in \mathbb{N},$$

for some constant $C \geq 0$. Prove that $f_n \to f$ a.e. as $n \to \infty$.

4. Assume that f is integrable on $\mathbb{R}$ with respect to the Lebesgue measure m and that $f(x) > 0$ a.e. Let E_k be a sequence of Lebesgue measurable sets in $[0,1]$ such that

$$\lim_{k\to\infty} \int_{E_k} f(x)\, dx = 0.$$

Prove that $\lim_{k\to\infty} m(E_k) = 0$.