COMPLEX ANALYSIS GRADUATE EXAM

Fall 2021

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_0^\infty \frac{dx}{x^{1/3}(1+x)},$$

carefully justifying all the steps.

2. Suppose that $\mathcal{F}$ is a normal family of analytic function on the open unit disk $\mathbb{D}$. Prove that $\mathcal{G} = \{f' : f \in \mathcal{F}\}$ is a normal family on $\mathbb{D}$.

3. Let $\Omega \subseteq \mathbb{C}$ be an open bounded connected set. Suppose that f is continuous on $\overline{\Omega}$, holomorphic in Ω, and that it satisfies $f(z) \neq 0$ for $z \in \overline{\Omega}$ and $|f| = C$ on $\partial\Omega$, where C is a constant.
(a) Prove that f is a constant function.
(b) Is the boundedness condition imposed on Ω essential?

4. Find all entire functions f such that $f((1+i)z) = f(z)$ for all $z \in \mathbb{C}$.