ALGEBRA QUALIFYING EXAM FALL 2021

PROBLEM 1. Classify all groups of order $13^2 \times 7$.

PROBLEM 2.

(a) If G is a transitive subgroup of S_n with $n > 1$ which has no fixed points, prove that there exists $g \neq 1$ in G so that g has no fixed points.

(b). Give an example of $G < S_n$ so that every element of G has fixed points but G itself has no fixed points..

PROBLEM 3. Let b be any integer coprime to 7 and consider the polynomial $f_b(x) = x^3 - 21x + 35b$. Show that f_b is irreducible over $\mathbb{Q}$. Write P for the set of $b \in \mathbb{Z}$ such that b is coprime to 7 and the Galois group of f_b is the alternating group. Find P. (Hint: the discriminant of the cubic polynomial of the form $x^3 + px + q$ is given by $-4p^3 - 27q^2$.)

PROBLEM 4. Suppose R is a commutative local ring so R has a unique maximal ideal $\mathfrak{m}$. Show if $x \in \mathfrak{m}$, then $1 - x$ is invertible. Show if, in addition, R is Noetherian and $\mathfrak{a} \subset R$ is an ideal such that $\mathfrak{a}^2 = \mathfrak{a}$, then $\mathfrak{a} = 0$.

PROBLEM 5. Let A be a finite dimensional central division algebra over a field F. Prove $[A, A] \neq A$.

PROBLEM 6. Let $K \subset F$ be a nontrivial finite field extension. Prove $F \otimes_K F$ is not a domain.