MATH 507a **QUALIFYING EXAM** **September 2021**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

(1) Consider the following 3 possible properties of a random variable X with characteristic function φ:

(i) X is *discrete*, that is, there exists a countable $A \subset \mathbb{R}$ with $X \in A$ a.s.;

(ii) X is a *lattice random variable*, that is, there exist $a \in \mathbb{R}$ and $b \neq 0$ such that $X \in \{a + jb : j \in \mathbb{Z}\}$ with probability 1;

(iii) φ is periodic.

(a) Suppose $|\varphi(t)| = 1$ for some $t \neq 0$. Show that X is lattice. HINT: If $\varphi(t) = e^{i\theta}$ for some t, θ, what do you know about the random variable $e^{-i\theta}e^{itX}$?

(b) Prove, or disprove with a counterexample: if φ is periodic then X is discrete.

(c) Prove, or disprove with a counterexample: if X is discrete then φ is periodic.

(2) Fix $R > 0$ and let $B(0, R)$ be the ball of radius R in $\mathbb{R}^2$ centered at the origin. Let $X_0 = 1$, and suppose that for $n \geq 1$, given X_n, X_{n+1} is uniform in $B(0, R|X_n|)$, independent of $X_1, \ldots, X_n$. In other words, the random variables $X_{n+1}/|X_n|$ are independent, each uniform in $B(0, R)$.

(a) Find the density of $|X_1|$ and find $E(\log |X_1|)$.

(b) Show that there exists L_R such that $\frac{\log |X_n|}{n} \to L_R$ a.s. HINT: Relate $|X_n|$ to the ratios $X_{j+1}/|X_j|$.

(c) Find R_0 such that $X_n \to 0$ a.s. if $R < R_0$, but not if $R > R_0$.

(3)(a) Let $\{X_n, n \geq 1\}$ be random variables and $\mu \in \mathbb{R}$, let $Z_n = \sqrt{n}(X_n - \mu)$, and suppose $Z_n \to Z$ in distribution where Z is standard normal $N(0, 1)$. Let g be differentiable function with $g'(\mu) \neq 0$. Show that $\sqrt{n}(g(X_n) - g(\mu)) \to g'(\mu)Z$ in distribution. HINT: First show this for a.s. convergence.

(b) Let $\xi_1, \xi_2, \ldots$ be iid uniform in [0,1]. Find μ_n, σ_n such that

$$\frac{\left(\prod_{i=1}^n \xi_i\right)^{1/n} - \mu_n}{\sigma_n} \to Z \quad \text{in distribution,}$$

where Z is standard normal. HINT: Use (a).

SOME ANTIDERIVATIVES THAT MAY BE USEFUL IN THE EXAM (proof not required, and not all are necessarily needed):

$$\int \log t \, dt = t \log t - t + C, \quad \int (\log t)^2 \, dt = t(\log t)^2 - 2t \log t + 2t + C, \quad \int t \log t \, dt = \frac{t^2}{2} \log t - \frac{t^2}{4} + C.$$