MATH 505a **QUALIFYING EXAM** **September 2021**

Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all four problems.

(1)(a) Let X be a non-negative random variable with finite expectation. Show that

$$\sum_{i=1}^{\infty} P(X \geq i) \leq E[X] < 1 + \sum_{i=1}^{\infty} P(X \geq i).$$

(b) Show that if X takes values only in $\{0, 1, \ldots, n\}$ for some n, then the first inequality in (a) in an equality:

$$\sum_{i=1}^{\infty} P(X \geq i) = E[X].$$

(c) Let M be the minimum value seen in 4 die rolls. Find $E[M]$. You don't need to simplify to one number, just get an expression in terms of numbers only.

(2) Suppose X and Y are independent continuous random variables with uniform distribution on $[0, 1]$.

(a) Find the density function of $X + 2Y$.

(b) Find the joint density function for $X - Y, X + Y$.

(3) Consider Bernoulli trials with success probability $p \in (0, 1)$. Let p_n be the probability of an odd number of successes in n trials.

(a) Express p_n in terms of p_{n-1}.

(b) Based on (a), for what value λ does $p_{n-1} = \lambda$ imply $p_n = \lambda$?

(c) Show that $\lim_n p_n = \lambda$, the value you found in (b). HINT: Write p_n as $\lambda + \epsilon_n$, for the λ you found in (b).