Numerical Analysis Preliminary Examination Fall 2021

September 2, 2021

Problem 1.
Given a function of time, $g \in C(0,1)$, consider the inverse filtering, or deconvolution, problem of determining a function of time $f \in L_2(0,1)$, that satisfies $Hf = g$, where the linear operator H is the convolution operator on $L_2(0,1)$ given by

$$(Hf)(t) = \int_0^t h(t-s)f(s)ds, \quad 0 \le t \le 1,$$

with the kernel, or filter, $h \in C(0,1)$ known and given. Let V be an n dimensional subspace of $L_2(0,1)$ with basis functions $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$, let $\{t_1, t_2, \ldots, t_m\}$ be specified with $0 \le t_1 < t_2 < \ldots < t_m \le 1$ and $m > n$, and let g and h denote respectively, the given function and kernel in $C(0,1)$.

(a) Formulate the problem of finding a least squares solution $f^{m,n}$ in the subspace $V \subset L_2(0,1)$ to the approximating discretized inverse filtering problem given by

$$(Hf^{m,n})(t_i) = g(t_i), \quad i = 1, 2, \ldots, m,$$

in the form $\min_{\beta \in \mathbb{R}^n} \| A^{m,n}\beta - b^m \|^2$. That is, what are the matrix $A^{m,n}$ and the vector b^m?

(b) What is a necessary and sufficient condition on the matrix $A^{m,n}$ for the least squares problem formulated in part (a) to have a unique solution?

(c) If the condition in part (b) is not satisfied, what is the solution of minimum norm to the least squares problem in part (a) in terms of the singular values of the matrix $A^{m,n}$?

(d) If we define the row vector of functions $\Phi^n \in L_2^n(0,1)$ by $\Phi^n = [\varphi_1, \varphi_2, \ldots, \varphi_n]$, then any $\varphi^n \in V \subset L_2(0,1)$ can be written as $\varphi^n = \Phi^n \beta$, for some vector $\beta \in \mathbb{R}^n$. Show that $\| \varphi^n \|^2_{L_2(0,1)} = \beta^T M^n \beta$ for some matrix $M^n \in \mathbb{R}^{n \times n}$, and show that the matrix M^n is positive definite and symmetric.

(e) Now let $\lambda > 0$ be given, and formulate the approximating discretized inverse filtering problem with Tychonov regularization of minimizing over V (or equivalently, over $\mathbb{R}^n$) the functional

$$J(\beta; \lambda) = \| A^{m,n}\beta - b^m \|^2 + \lambda \| \varphi^n \|^2_{L_2(0,1)}$$

as a least squares problem of the form $\min_{\beta\in\mathbb{R}^n} \| \hat{A}^{m,n}(\lambda)\beta - \hat{b}^{m,n} \|^2$ (i.e. what are the matrix $\hat{A}^{m,n}(\lambda)$ and vector $\hat{b}^{m,n}$?) where $A^{m,n}$ and b^m are as they were defined in part (a) and φ^n and β are as they were defined in part (d).

(f) Verify that the least squares problem defined in part (e) has a unique solution and then find it.

Problem 2. Consider the vector space $\mathbb{R}^n$ endowed with the l_2-norm $||\cdot||_2$ and let $||\cdot||$ be the induced norm on $n\times n$ matrices. Let A be an $n\times n$ invertible matrix.

(a) Explain how we know that $\inf_{||v||_2=1}||Av||_2$ is attained and denote the unit vector that attains it by x.

(b) Let $y = Ax$. Show $||A^{-1}|| = \frac{1}{||y||_2}$.

(c) Give a geometric description of the action of the matrix yx^T and find $||yx^T||$. *Hint: what is the image of a vector $z = \alpha x + w$ where w is orthogonal to x?*

(d) Consider the matrix $B = A - yx^T$. Show B is singular.

(e) Let
$$\Delta = \inf\{||\delta A|| : A + \delta A \text{ is singular}\}$$
denote the distance of A from the set of singular matrices. Show
$$\Delta = \frac{1}{||A^{-1}||}.$$

Problem 3. Suppose A is a real $n\times n$ symmetric and positive definite matrix with eigenvalues
$$0 < \lambda_n \le \lambda_{n-1} \le \ldots \le \lambda_1$$
and $b\in\mathbb{R}^n$. The Richardson Iteration Method for finding the solution to $Ax = b$ is
$$x_{k+1} = x_k - \omega(Ax_k - b)$$
where ω is an iteration parameter.

(a) For what values of ω is the Richardson Iteration Method guaranteed to converge for any starting iterate x_0?

(b) Find the value of ω that optimizes the rate of convergence.

(c) Consider the matrix
$$A = \begin{bmatrix} 2+a & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2+a & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2+a & -1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2+a & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2+a \end{bmatrix}$$

where $a > 0$. Use Gerschgorin's Theorem to find upper and lower bounds for the eigenvalues of A.

(d) If you want to solve $Ax = b$ using the Richardson Iteration Method for the matrix A in (c), use your answers to (b) and (c) to determine a good choice for ω. Explain and give a bound for the number of iterations sufficient to reduce the norm of the matrix by a factor of 10^{-6}.

Problem 4.

(a) Consider a matrix that has real eigenvalues and n linearly independent eigenvectors x_i, and the largest eigenvalue in magnitude λ_1 is dominant. That is ($|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$). Show that if the vector v_0 has a nonzero component α_1 in the direction of x_1 then $\lim_{m\to\infty}\left(\frac{1}{\lambda_1^m}\right)A^m v_0 = \alpha_1 x_1$

Also, $\lambda_1 = \lim_{m\to\infty}\frac{y^T A^{m+1} v_0}{y^T A^m v_0}$ where y is a vector not orthogonal to x_1.

(b) Get an approximation for x_1 and λ_1 by doing three iterations of the method in part (a). Take $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and start from $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Comment on your method and results.

(c) A Householder transformation is a matrix of the form:

$$H = I - 2\frac{vv^T}{||v||^2}.$$

Show that H is symmetric and orthogonal. Furthermore, show that if

$$z = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \sigma = ||x||, \ v = x + \sigma\ z \text{ Then } Hx = -\sigma\ z$$

(d) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Using part (c) find an orthogonal matrix U such that $U^{-1}AU$ is tridiagonal. How can this help us to compute the eigenvalues of A?. Check that $U^{-1}AU$ is tridiagonal. Hint: The matrix H will be imbedded into the lower right corner of the matrix U