ODE EXAM - Fall 2020

The exam has **four** problems. Each problem is worth 10 points. Do all four problems.

1. Let ε be a constant, and consider the differential equation

$$y'(t) = -2y(t) + \varepsilon \sin\big(y(t)\big),\ t > 0.$$

Determine, with proof, the largest possible value of the number $a > 0$ such that, for every $\varepsilon \in [-a, a]$, all solutions of the equation satisfy $\lim\limits_{t\to+\infty} |y(t)| = 0$.

2. Consider the linear system of ODEs in $\mathbb{R}^2$,

$$\begin{bmatrix} x'(t) \\ \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{1}{1+|t|} & \dfrac{1}{1+t^2} \\ \\ t^2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ \\ y(t) \end{bmatrix}.$$

Show that there exists at least one solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

such that

$$\lim_{t\to+\infty} \Big(|x(t)| + |y(t)|\Big) = +\infty.$$

3. For the nonlinear system

$$\begin{cases} x' = -y - x(x^2+y^2)^3 \\ y' = \quad x - y(x^2+y^2)^3, \end{cases}$$

determine the location and type of all critical points.

4. Let σ, r and b be positive constants, and consider the Lorentz system

$$\begin{cases} x' = -\sigma(x-y) \\ y' = rx - y - xz \\ z' = -bz + xy. \end{cases}$$

(a) Prove that if $r > 1$, then the origin $(0, 0, 0)$ is a hyperbolic fixed point of saddle type.

(b) Write the equation of the tangent plane to the stable manifold of the system at the origin when $b = 2, r = 4, \sigma = 1$.