PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Fall 2020

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $u(x)$ be a smooth function satisfying

$$\begin{aligned} \Delta u(x) &= f(u(x)), \quad x \in B(0,1) \\ u(x) &= 0, \quad x \in \partial B(0,1) \end{aligned}$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^n$ and f is continuous.

(a) Show that if $f(t)$ has the same sign of t, then u must be identically zero in $B(0,1)$.

(b) What can you say about the solution when $f(t) = t^4$?

2. Suppose $u = u(t,x)$ is smooth and bounded, and solves the nonlinear heat equation

$$\begin{aligned} (\partial_t - \Delta)u &= |\nabla u|^2, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) &= f(x), \quad x \in \mathbb{R}^n \end{aligned}$$

(a) Prove that $v = e^u$ solves the linear heat equation $(\partial_t - \Delta)v = 0$.

(b) Find an explicit formula for u in terms of f.

3. Suppose $u = u(t,x)$ solves the following initial-boundary value problem

$$\begin{aligned} (1+t)\partial_t^2 u - \Delta u + \partial_t u &= 0, \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u(t,x) &= 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ u(0,x) &= u_0(x), \quad x \in \Omega \\ \partial_t u(0,x) &= u_1(x), \quad x \in \Omega \end{aligned}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$, and $u_0(x)$ and $u_1(x)$ are smooth and compactly supported in Ω.

(a) Show that the L^2 norm of the solution u is bounded for all $0 < t < \infty$.

(b) What can you say about the L^2 norm of $\partial_t u$ as $t \to \infty$?