1. Let F be a given strictly increasing cumulative distribution function with corresponding density f on the real line. Assume that n i.i.d. random variables are generated from F, but we are told only $X := \max(X_1, \dots, X_n)$. In particular, the positive integer n itself is unknown.

   (a) Find the probability density function $f_n(\cdot)$ of X.

   (b) Show that the family $\{f_n(\cdot),\ n \geq 1\}$ has monotone likelihood ratio with respect to some statistic, and make sure to identify the statistic involved.

   (c) Find the uniformly most powerful test for testing $H_0 : n \leq 5$ against $H_a : n > 5$.

2. Suppose that $X_1, \dots, X_n$ are i.i.d. random variables with mean μ, variance σ^2 and finite moments of all orders. We are interested in estimating $g(\mu)$ where $g : \mathbb{R} \mapsto \mathbb{R}$ is some smooth function with bounded third derivative.

   (a) The "plug-in" estimator of $g(\mu)$ is $g(\overline{X}_n)$ where $\overline{X}_n = \frac{1}{n}\sum_{j=1}^n X_j$ is the sample mean. Using Taylor's expansion and assuming that n is large, find the leading term in the expression for the bias $b := Eg(\overline{X}_n) - g(\mu)$ of $g(\overline{X}_n)$ for estimating $g(\mu)$.

   (b) Since μ is unknown, the bias is also unknown. Explain how one can use the non-parametric bootstrap to estimate the bias b. Specifically, write down the expression for the bootstrap estimator of the bias.

   (c) Let $g(x) = x^2$ and $n = 2$. Suppose that the observed values are $X_1 = 1$ and $X_2 = 3$. Find the exact value of the bootstrap estimator of the bias in this case. Which value would you use to estimate $g(\mu)$?