1. For $\lambda > 0$, consider observing a single observation $X \sim \mathcal{P}(\lambda)$ from the Poisson distribution, satisfying

$$P_\lambda(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad k = 0, 1, 2 \ldots. \tag{1}$$

(For instance, X might be the number of arrivals in the time interval $[0, 1]$, when arrivals follow a Poisson process with rate λ.)

   (a) Based on X, find the UMVU of $\phi(\lambda) = e^{-3\lambda}$ (the probability that there are no arrivals in the interval $[1, 4]$). Hint: Find a function g that satisfies

$$E_\lambda[g(X)] = e^{-3\lambda},$$

   using (1) and the infinite series representation of the exponential function. Does the UMVU exist uniquely? Compute the variance of the resulting estimator.

   (b) Compute the value of the estimator for some small values of X, and comment on any peculiarities you observe.

2. Let $X_1, \ldots, X_n$ be a sample from the $\theta = \theta_0$ density from the family $\{p(x; \theta), \theta \in \Theta\}$ of positive densities, where $\Theta = \{\theta_0, \theta_1, \ldots, \theta_d\}$ is a finite set, with the distributions corresponding to differing parameters being unequal.

   (a) Define the scaled log likelihood function

$$\ell_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \log p(X_i; \theta).$$

   Making any necessary first moment assumptions, use the law of large numbers to identify the limit for all $\theta \in \Theta_0$, and prove convergence to it; make sure to state what type of convergence is claimed to hold.

   (b) For p and q any density functions, recall the Kullback-Leibler divergence

$$D(P||Q) = E_p\left[\log \frac{p(Y)}{q(Y)}\right],$$

   where E_p means we take expectation with respect to density p. Use Jensens's inequality to show that $D(P||Q) \geq 0$. Is it the case that the inequality will be strict when the distributions P and Q are not equal?

   (c) Define the maximum likelihood estimate as

$$\widehat{\theta}_n = \operatorname{argmax} \ell_n(\theta),$$

   that is, as the θ value that achieves the maximum log likelihood for parameters in Θ; in the case of a tie, the parameter θ_j with the highest index j shall be chosen. Prove that $\widehat{\theta}_n$ converges to the true value θ_0. If convergence occurs in multiple ways, prove the strongest statement you can.