Geometry and Topology Graduate Exam

Fall 2020

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Suppose that the space X admits a universal covering space $\widetilde{X}$, namely a covering space that is path connected and simply connected. Show that, if $\widetilde{X}$ is compact, the fundamental group of X is finite.

Problem 2. Let X be the topological space obtained from a regular $2n$-gon by identifying opposite edges with parallel orientations. Write a presentation for its fundamental group $\pi_1(X; x_0)$, for a base point $x_0 \in X$ of your choice. (The answer, as a function of n, may depend on the parity of n.)

Problem 3. In $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, consider the subset

$$X = \left(S^1 \times S^1\right) \cup \left(\{x_0\} \times B^2\right) \cup \left(B^2 \times \{x_0\}\right)$$

where the disk B^2 is bounded by the unit circle S^1 in $\mathbb{R}^2$, and where $x_0 \in S^1$. Compute the homology group $H_p(X; \mathbb{Z})$ for all p.

Problem 4. In the vector space $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of all n-by-n matrices, let $\mathrm{SL}_n(\mathbb{R})$ be the special linear group consisting of all $A \in \mathrm{M}_n(\mathbb{R})$ with $\det A = 1$. For $A \in \mathrm{SL}_n(\mathbb{R})$, describe the tangent space $T_A\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ by an explicit equation. Possible hint: begin with the case where A is the identity matrix I_n.

Problem 5. Consider the 1–form $\lambda \in \Omega^1(M)$ and the 2-form $\omega = d\lambda \in \Omega^2(M)$ on the manifold M. Suppose that L is a submanifold of M and that, for the inclusion map $i\colon L \to M$, the pull-back $i^*(\lambda) \in \Omega^1(L)$ is exact, in the sense that there exists a function $\varphi\colon L \to \mathbb{R}$ such that $i^*(\lambda) = d\varphi$. Show that, for the unit disk $D^2 \subset \mathbb{R}^2$ and for any smooth map $f\colon D^2 \to M$ which sends the boundary of the disc to L,

$$\int_{D^2} f^*(\omega) = 0.$$

Problem 6. Let $f\colon S^1 \times S^1 \to S^1 \times S^1$ be the map that, identifying S^1 with the unit circle in the complex plane $\mathbb{C}$, is defined by

$$f(z_1, z_2) = \left(z_1^2 z_2, z_1^{-1} z_2\right)$$

for every z_1, $z_2 \in S^1 \subset \mathbb{C}$. Compute the homomorphism $H^2(f)\colon H^2_{\mathrm{dR}}(S^1 \times S^1) \to H^2_{\mathrm{dR}}(S^1 \times S^1)$ induced by f on the de Rham cohomology space $H^2_{\mathrm{dR}}(S^1 \times S^1)$. Hint: what is the degree of f?

Problem 7. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ be a 2-form on $\mathbb{R}^4$ with standard coordinates x_1, x_2, x_3, x_4. Consider the vector field $Z = 3x_1\partial_{x_1} + 3x_2\partial_{x_2} + 3x_3\partial_{x_3} + 3x_4\partial_{x_4}$, and let $(\varphi_t)_{t\in\mathbb{R}}$ be the flow that it defines; you may take for granted that this flow exists for all time t. Calculate the pull back $(\varphi_t)^*\omega$. Hint: look at the differential equation that $(\varphi_t)^*\omega$ satisfies.