REAL ANALYSIS GRADUATE EXAM

Fall 2020

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f_n be a sequence of measurable functions on (X, μ) and $f_n \geq 0$. Assume that $\int f_n \, d\mu = 1$ for all $n \in \mathbb{N}$. Prove that $\limsup_{n\to\infty} f_n(x)^{1/n} \leq 1$ for μ-a.e. x.

2. Let $x = 0.n_1 n_2 n_3 \ldots$ be the decimal expansion of $x \in [0, 1]$, and define the function $f(x) = \min_i n_i$. Prove that f is measurable and constant a.e. (When a number has two different representations, we use the one with repeated zeros.)

3. Let f be a continous function on $[0, 1]$. Let $F(x) = \sup_{0\leq y\leq x} f(y)$ for $x \in [0, 1]$.
a) Is F a Borel function?
b) Prove that $S = \{x \in (0, 1] : f(x) > f(y) \text{ for all } 0 \leq y < x\}$ is Borel.

4. Let $f(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$ for $x \geq 0$, where $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $\sum_{n=1}^{\infty} n a_n < \infty$ if and only if the right-hand derivative of f exists at 0.