COMPLEX ANALYSIS GRADUATE EXAM

Fall 2020

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Compute

$$\int_{-\infty}^{\infty} \frac{e^{-itx}}{a^2+x^2}\, dx,$$

where $a > 0$ and $t \in \mathbb{R}$, justifying all the steps.

2. Let $f(z)$ be a holomorphic function on the unit disk $D = \{z : |z| < 1\}$, and $f(0) = 0$. Show that $F(z) = \sum_{n=1}^{\infty} f(z^n)$, for $z \in D$, is holomorphic on D.

3. Is there a holomorphic function f on the unit disc so that $m_n \to \infty$ as $n \to \infty$, where

$$m_n = \inf\left\{ |f(z)| : 1 - \frac{1}{n} < |z| < 1 \right\}?$$

4. Denote by $D(0, r)$ the open disc around 0 with radius r. Let f, g be holomorphic functions from $D(0, 1)$ into an open set Ω. Assume that f is one-to-one, $f(D(0, 1)) = \Omega$ (thus f is bijective from $D(0, 1)$ to Ω), and $f(0) = g(0)$. Prove that

$$g(D(0, r)) \subseteq f(D(0, r)),$$

for all $r \in (0, 1)$.