ALGEBRA QUALIFYING EXAM FALL 2020

PROBLEM 1. Let V_A be a $\mathbb{Q}[x]$-module that corresponds to a matrix $A \in \mathrm{Mat}_n(\mathbb{Q})$. In other words, $V_A = \mathbb{Q}^n$, and the $\mathbb{Q}[x]$-module structure is defined by $f \cdot v = f(A)v$, $f \in \mathbb{Q}[x]$, $v \in \mathbb{Q}^n$. The module V_A is called cyclic if there is $v \in V$ so that $\mathbb{Q}[x] \cdot v = V_A$. The matrix A is called cyclic if V_A is cyclic.

(1) Prove: V_A is cyclic if and only if $\mathrm{Ann}(V_A)$ is generated by the characteristic polynomial $\chi(A)$.

(2) For any matrix A there is a cyclic matrix C such that $AC - CA = 0$.

PROBLEM 2. A finite dimensional algebra over a field has finitely many up to isomorphism simple left modules.

PROBLEM 3. Find all pairwise non-isomorphic groups of order 147 that contain no elements of order 49.

PROBLEM 4. Prove that if for $f, g \in \mathbb{C}[x, y]$ the system of equations $f = g = 0$ has finitely many solutions, then the algebra $\mathbb{C}[x, y]/(f, g)$ is finite dimensional.

PROBLEM 5. Show that a finitely generated projective module over a principal ideal domain is free.

PROBLEM 6. Show that $p(x) = x^5 - 4x + 2$ is irreducible over $\mathbb{Q}$, and that it has exactly three real roots. Use this to show that the Galois group of $p(x)$ is S_5. Answer the question: Is $p(x) = 0$ solvable by radicals?