MATH 507a **QUALIFYING EXAM** **September 2020**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

(1) Let $X_1, X_2, \dots$ be i.i.d. nonnegative random variables with $P(X_1 > 0) > 0$.

(a) Show that $\limsup_n X_n^{1/n} \geq 1$ a.s. HINT: Consider events $\{X_n^{1/n} \geq 1 - \epsilon \text{ i.o.}\}$.

(b) Let $c > 1$. Show that $P(X_n^{1/n} \geq c \text{ i.o.}) = 1$ if and only if $E(\log^+ X_1) = \infty$. (Here $\log^+ x$ means $\log x$ if $\log x > 0$, and 0 otherwise.) HINT: $E(\log^+ X_1)$ is an integral, but it can be compared to a sum in a standard way. Also, for $t > 0$, the statement $\log^+ x > t$ is the same as $\log x > t$.

(c) Show that the only possible values of $\limsup_n X_n^{1/n}$ are 1 and ∞.

(2) Suppose $X_1, X_2, \dots$ are iid with characteristic function $\varphi(t) = 1 - \beta|t|^\alpha + o(|t|^\alpha)$ as $t \to 0$. Let Z have characteristic function $\psi(t) = e^{-|t|^\alpha}$ for some $\alpha \in (0, 2]$. Find b, θ (expressed in terms of α and β) so that $S_n / b n^\theta$ converges weakly to Z.

(3) Suppose X, Y are r.v.'s with $E(X^2) < \infty$, and let

$$\mathcal{H} = \{ \text{ all measurable } h : \mathbb{R} \to \mathbb{R} \text{ for which } E(h(Y)^2) < \infty \ \}.$$

(a) For $h \in \mathcal{H}$ show that $E\big[\big(X - E(X \mid Y)\big)\big(E(X \mid Y) - h(Y)\big)\big] = 0$.

(b) Show that the choice of h which minimizes $E[(X - h(Y))^2]$ over $h \in \mathcal{H}$ is $h(y) = E(X \mid Y = y)$, that is, $h(Y) = E(X \mid Y)$. HINT: Use (a).