Numerical Analysis Preliminary Examination Fall 2020

September 9, 2020

Problem 1.
Consider the normed linear space $\mathbb{C}^n$ with vector norm $\| \cdot \|$ and let $A, B \in \mathbb{C}^{n\times n}$. The definition of the matrix norm, $\| \cdot \|_M$, induced by the vector norm $\| \cdot \|$ is given by $\| A \|_M = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$.

(a) Show that for each $A \in \mathbb{C}^{n\times n}$ and each $x \in \mathbb{C}^n$, $\| Ax \| \leq \| A \|_M \| x \|$.

(b) Show that $\| A \|_M = \sup_{\|x\|\leq 1} \| Ax \|$.

(c) Show that in fact for each $A \in \mathbb{C}^{n\times n}$ there exists a $y \in \mathbb{C}^n$ with $\| y \| \leq 1$
such that $\| A \|_M = \| Ay \|$.

(d) Show that $\| AB \|_M \leq \| A \|_M \| B \|_M$.

(e) If $\| \cdot \| = \| \cdot \|_1$ find $\| \cdot \|_M$.

Problem 2.
Let $A \in \mathbb{C}^{m\times n}$ have full column rank.

(a) Find the mapping, P, of $\mathbb{C}^m$ onto the range of A defined by

$$Px = \operatorname{argmin}_{y\in \text{Range}(A)} \| x - y \|_2^2 \quad x \in \mathbb{C}^m.$$

Hint: Use the normal equations.

(b) Show that P is self-adjoint and idempotent.

(c) Let $C, D \in \mathbb{C}^{m\times m}$, let $C = U\Sigma V^*$ be a singular value decomposition for C and suppose that $C = (I - 2P)D(I - 2P)^*$. Find a singular value decomposition for D. Justify your answer.

Problem 3.

(a) Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ \delta & 1 \end{bmatrix}.$$

Show that for any $\varepsilon > 0$ and any real number K there exists δ such that $||A-B||_2 < \varepsilon$ and the eigenvalues μ of B have the property that for all eigenvalues λ of A,

$$|\mu - \lambda| \geq K||A-B||_2.$$

(b) Suppose now A is a real $n \times n$ symmetric matrix with eigenvalues λ_1, λ_2, …, λ_n and suppose $\mu \in \mathbb{R}$ and $u, w \in \mathbb{R}^n$ have the property that

$$(A - \mu I)u = w.$$

Show there is at least one eigenvalue λ_j such that

$$|\mu - \lambda_j| \leq \frac{||w||_2}{||u||_2}.$$

Hint: expand u and w in an appropriate basis.

(c) Use your result in b) to show that if A is a real symmetric matrix with eigenvalues λ_1, λ_2, …, λ_n and B is a perturbation of A that is also symmetric, then every eigenvalue μ of B has the property that there is at least one eigenvalue λ_j such that

$$|\mu - \lambda_j| \leq ||A-B||_2.$$

Hint: Let u be a corresponding eigenvector of B and write $(A-\mu I)u$ as $(A-B+B-\mu I)u$.

(d) Consider the matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Suppose this matrix is entered into the computer as

$$B = \begin{bmatrix} 1.0000 & 0.5000 & 0.3333 \\ 0.5000 & 0.3333 & 0.2500 \\ 0.3333 & 0.2500 & 0.2000 \end{bmatrix}$$

whose eigenvalues are

$$\mu_1 = 1.408294053$$
$$\mu_2 = 0.11223414532$$
$$\mu_3 = 0.002664493933$$

What can we say about where the eigenvalues of A lie?

Problem 4.

(a) Let Q be a symmetric matrix. Show that any two eigenvectors of Q, corresponding to distinct eigenvalues, are Q-conjugate.

(b) Let Q be a positive definite symmetric matrix and suppose $p_0, p_1, \ldots, p_{n-1}$ are linearly independent vectors in $\mathbb{R}^n$. Show that the Gram-Schmidt procedure can be used to generate a sequence of Q-conjugate directions from the p_i's. Specifically, show that $d_0, d_1, \ldots, d_{n-1}$ defined recursively by

$$d_0 = p_0$$
$$d_{k+1} = p_{k+1} - \sum_{i=0}^{k} \frac{p_{k+1}^T Q d_i}{d_i^T Q d_i} d_i$$

forms a Q-conjugate set.

(c) Let Q be a positive definite symmetric matrix and suppose p_0, p_1, ..., p_{n-1} are linearly independent vectors in $\mathbb{R}^n$. Show how to find a matrix E such that $E^T Q E$ is diagonal.