ODE EXAM - Fall 2019

The exam has **five** problems on **two** pages. Each problem is worth 10 points. Do all five problems.

1. Consider the 2^{nd} order ODE,
$$x'' + p(t)x' + ax = 0,$$
where a is a real number and the function $p = p(t),\ t \geq 0$, is continuous and satisfies
$$\lim_{t\to+\infty} \int_0^t p(s)ds = +\infty.$$
Let $\phi(t)$ and $\psi(t)$ be a fundamental set of solutions of the above equation so that the matrix
$$X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}$$
is non-singular for all $t \geq 0$. Prove $\lim_{t\to+\infty} \det X(t) = 0$, where $\det A$ denotes the determinant of the matrix A.

2. Consider the planar system of differential equations
$$\begin{aligned} x' &= -y + x\big[(x^2+y^2)^2 - 3(x^2+y^2) + 1\big], \\ y' &= \ \ x + y\big[(x^2+y^2)^2 - 3(x^2+y^2) + 1\big]. \end{aligned}$$
   (a) Prove that the system has a periodic orbit in the annular region
$$D = \{(x,y) \in \mathbb{R}^2;\ 1 < x^2+y^2 < 3\}.$$
   (b) What can we conclude about the index of this orbit?

   (c) What can we conclude about the number of stationary points in D?

   Give your reasons and details.

3. Consider the linear system of ODEs in $\mathbb{R}^n$,
$$x' = A(t)x, \tag{1}$$
where $A = A(t)$ is an $n \times n$ matrix-valued function, continuous for all $t \in \mathbb{R}$.

   Assume that there exists a number $T > 0$ such that $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Describe the structure of a fundamental solution matrix of (1). Justify your conclusions.

4. Once again, consider the system (1), but this time assume that there is a real number δ such that the eigenvalues $\{\lambda_1(t), \ldots, \lambda_n(t)\}$ of the matrix $A(t)$ satisfy

$$\mathrm{Re}\ \lambda_j(t) \leq \delta,$$

for all $j = 1, \ldots, n$ and all $t \geq 0$; Re z denotes the real part of the complex number z. If it helps, you can keep assuming that the matrix A is periodic.

(a) True or false: if $\delta = 0$ then every solution $x = x(t)$ of (1) satisfies $\sup_{t>0} \|x(t)\| < \infty$? Explain your conclusion.

(b) True or false: if $\delta < 0$ then every solution $x = x(t)$ of (1) satisfies $\lim_{t\to+\infty} \|x(t)\| = 0$? Explain your conclusion.

(c) True or false: if $\delta < 0$ then every solution $x = x(t)$ of (1) satisfies $\sup_{t>0} \|x(t)\| < \infty$? Explain your conclusion.

(d) Will your answers to the above three questions change if the matrix $A = A(t)$ is symmetric for all $t \geq 0$? Explain your conclusion.

5. Let $f = f(x)$, $x \in \mathbb{R}$, be a continuously differentiable function such that the equation $x' = f(x)$ has no stationary solutions. Prove that if $x = x(t)$ is a solution of the equation and $x(t)$ is defined for all $t \geq 0$, then

$$\lim_{t\to+\infty} |x(t)| = +\infty.$$