PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Fall 2019

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Suppose f is a compactly supported smooth function on $\mathbb{R}^3$. Prove that there is a unique smooth function u on $\mathbb{R}^3$, such that

$$-\Delta u = f, \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0.$$

For this u, find the value of

$$\lim_{|x|\to\infty} |x|u(x).$$

2. Consider the following one-dimensional heat equation:

$$\begin{cases} \partial_t u = \partial_x^2 u, & (t,x) \in (0,+\infty) \times (0,1), \\ u = 0, & (t,x) \in (0,+\infty) \times \{0,1\}. \end{cases}$$

Find all solutions that have factorized form $u(t,x) = \alpha(t)\beta(x)$.

3. Suppose u solves the following initial-boundary value problem:

$$\begin{cases} \partial_t^2 u = \partial_x^2 u - u^3, & (t,x) \in (0,+\infty) \times (0,1), \\ u = 0, & (t,x) \in (0,+\infty) \times \{0,1\}, \\ u(0,x) = u_0(x), & x \in (0,1), \\ \partial_t u(0,x) = u_1(x), & x \in (0,1), \end{cases}$$

where u_0, u_1 are smooth functions.

(a) Find an energy $E(t)$ which is independent of t.

(b) Show that u is bounded for all (t,x), namely $|u(t,x)| < C$ for some constant C for all $(t,x) \in (0,+\infty) \times (0,1)$.