1. Let $X_1, \ldots, X_n$ be i.i.d. with exponential distribution with parameter $\theta > 0$, meaning that that pdf of X_1 is $f_\theta(x) = \begin{cases} \theta e^{-x\theta}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

   (a) Find the Maximum Likelihood estimator of θ.

   (b) Find the Likelihood ratio test for testing $H_0 : \theta = \theta_0$ against the alternative $H_a : \theta \neq \theta_0$ to make the test approximately level $\alpha \in (0, 1)$. Specify the test statistic and the rejection region (based on its asymptotic distribution).

   (c) Show that the Likelihood ratio test for testing $H_0 : \theta \leq \theta_0$ against the alternative $H_a : \theta > \theta_0$ (where $\theta_0 > 0$) rejects when $\bar{X}_n \leq C$ for C depending on the desired size of the test, where $\bar{X}_n = \frac{1}{n}\sum_{j=1}^n X_j$.

2. Let X_1 and X_2 be two independent observations from some distribution F with mean θ (nothing else is known about F). Assume that we estimate the mean via the sample mean $\hat{\theta} = \frac{X_1+X_2}{2}$. We are interested in estimating the probability $\lambda(t) = P_F\left(\hat{\theta} - \theta \leq t\right)$ for all t.

   (a) We are going to estimate $\lambda(t)$ using bootstrap. Assume that observed values are $X_1 = 1$ and $X_2 = 3$. Let $\hat{X}_1$ and $\hat{X}_2$ be the bootstrap sample, and find its distribution (for better readability, you may draw a small table with possible values of $(\hat{X}_1, \hat{X}_2)$ and their probabilities).

   (b) Let $\hat{\theta}^* = \frac{\hat{X}_1+\hat{X}_2}{2}$. Find the distribution of $\hat{\theta}^* - \hat{\theta}$ (conditionally on the values of X_1 and X_2), and estimate $\lambda(-0.5)$.

   (c) Now assume that F has density $f_\theta(x) = \frac{1}{2}e^{-|x-\theta|}$. Again, we would like to estimate the probability $P_F\left(\hat{\theta} - \theta \leq t\right)$ via bootstrap. How would you proceed in this case? Your solution has to use the additional information about the density.