Geometry and Topology Graduate Exam
Fall 2019

Solve all seven problems. Every problem is weighted equally. Partial credit will be given to partial solutions.

Problem 1. Let X be the union of the twelve edges of a regular cube in $\mathbb{R}^3$. Compute the fundamental group of the complement $\mathbb{R}^3 - X$.

Problem 2. Let $\mathbb{CP}^1$ denote the complex projective space of dimension 1, and let $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ be the map induced by the polynomial

$$P(X) = X^7 + 5X^3 - 6X^2 + 1.$$

(So, f sends the line passing through $(x, 1) \in \mathbb{C}^2$ to the line passing through $(P(x), 1)$.) If $\alpha \in \Omega^2(\mathbb{CP}^1)$ is a differential form of degree 2 and we write

$$K := \int_{\mathbb{CP}^1} \alpha,$$

compute the integral

$$\int_{\mathbb{CP}^1} \Omega^2(f)(\alpha)$$

of the pullback $\Omega^2(f)(\alpha) \in \Omega^2(\mathbb{CP}^1)$ of α under f in terms of K. (Possible hint: degree.)

Problem 3. Let X and Y be two topological spaces and let $f, g : X \to Y$ be two continuous maps. Consider the topological space

$$Z = \Big(Y \bigsqcup (X \times [0,1])\Big) \Big/ \begin{array}{l}(x,0) \sim f(x)\\ (x,1) \sim g(x)\end{array}$$

obtained from the disjoint union $Y \bigsqcup (X \times [0,1])$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form

$$\cdots \longrightarrow H_{n+1}(Z) \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \ldots\ ,$$

and identify the homomorphisms involved.

Problem 4. Let $f\colon S^n \to S^n$ be a continuous map that has no fixed points. Find the degree of f. (Hint: $a(x) = -x$.)

Problem 5. Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonzero polynomial over $\mathbb{R}$ in n variables that is homogenous of degree d (i.e. $p(\lambda \cdot \vec{x}) = \lambda^d \cdot p(\vec{x})$ for all $\lambda \in \mathbb{R}$). Show that $p^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ for all $c \neq 0$.

Problem 6. Define a differential 1-form on the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$$

which is closed but not exact (and show that it has these properties).

Problem 7. Prove that the subset

$$M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1^2 + x_2^2 + x_3^2 = x_4^3 + x_5^3\}$$

is not a submanifold of $\mathbb{R}^5$. (Hint: $(0, 0, 0, 0, 0)$.)