Algebra Qualifying Exam - Fall 2019

1. Suppose p and q are primes with $p < q$. If $n \geq 0$ is an integer, then show that any finite group G of order pq^n is solvable.

2. Let G be a finite group, $H \subset G$ a subgroup and S a Sylow p-subgroup of G.
   (a) Show that the intersection of H with some conjugate of S is a Sylow p-subgroup of H.
   (b) Give an example to show that $H \cap S$ need *not* be a Sylow p-subgroup of H.

3. Give an example of a field extension of degree 4 that has no intermediate subfield of degree 2 (hint: consider a Galois extension with group S_4).

4. Prove that the subset $\{(u^3, u^2v, uv^2, v^3), u, v \in \mathbb{C}\} \subset \mathbb{C}^4$ is algebraic.

5. Let k be a field.
   (a) Prove that if $A, B \in M_n(k)$ are 3×3 matrices, then A and B are similar if and only if they have the characteristic and minimal polynomials.
   (b) Show the statement in the preceding point may fail for 4×4 matrices.

6. If R is a left Noetherian ring, then show that every element $a \in A$ that admits a left inverse actually admits a 2-sided inverse.