MATH 507a **QUALIFYING EXAM** **August 2019**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Suppose $g : \mathbb{R} \to [0, \infty)$ is continuous, with $g > 0$ and $x^2/g(x) \to 0$ as $|x| \to \infty$. Suppose $X_n \implies X$ and $\sup_n E[g(X_n)] < \infty$. Show that $E[X_n^2] \to E[X^2]$. HINT: Truncate the function $h(x) = x^2$ at $\pm A$ to make a new function $h_A(x)$, for some A.

(2) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-field. Suppose $E(X^2) < \infty$.
(a) Show that $E(XE(X \mid \mathcal{G})) = E(E(X \mid \mathcal{G})^2)$.
(b) Suppose $EX^2 = E(E(X \mid \mathcal{G})^2)$. Show that $X = E(X \mid \mathcal{G})$ a.s. HINT: $X = (X - E(X \mid \mathcal{G})) + E(X \mid \mathcal{G})$.

(3) Suppose $X_1, X_2, \dots$ are iid with distribution function $F(x) = 1 - e^{-2x^{1/2}}, x \geq 0$.
(a) Find $\alpha_n \nearrow \infty$ for which $\limsup_n X_n/\alpha_n = 1$ a.s.
(b) Let $M_n = \max(X_1, \dots, X_n)$. Find c such that $\limsup_n M_n/\alpha_n = c$ a.s. (Prove your answer.)