MATH 505a **QUALIFYING EXAM** **August 2019**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Suppose A, B, C are pairwise independent, $A \cap B \cap C = \emptyset$, and $P(A) = P(B) = P(C) = p$.

(a) What is the largest possible value of p?

(b) Is it possible that $P(A \cup B \cup C) = 1$? Prove or disprove.

(2) Consider two coins: coin 1 shows heads with probability p_1 and coin 2 shows heads with probability p_2. Each coin is tosses repeatedly. Let T_i be the time of first heads for coin i, and define the event $A = \{T_1 < T_2\}$.

(a) Find $P(A)$. HINT: One possible method is to condition on one of the variables.

(b) Find $P(T_1 = k \mid A)$ for all $k \geq 1$.

(3) Players A and B are having a table tennis match; the first player to win 3 games wins the match. One of the players is better than the other; this better player wins each game with probability 0.7. Carl comes to watch the match. He does not know who is the better player so (based on Carl's information) A, B each initially have probability 0.5 to be the better player. Then Carl sees A win 2 of the first 3 games.

(a) What is now the probability (after the 3 games, based on Carl's information) that A is the better player? Simplify your answer to a single fraction or decimal.

(b) What is now the probability (after the 3 games, based on Carl's information) that A will go on to win the match?

NOTE: Express your answer for (b) in terms of numbers; you do not need to simplify to a single number. An answer in a form like $\frac{5}{4} + 7(2 - \frac{9}{5})$ is OK.

(4) Suppose X_n is binomial with parameters (n, p) with $0 \leq p \leq 1$, and X is Poisson(λ).

(a) Find the moment generating function of X_n.

(b) Suppose $n \to \infty$ and $p = p_n \to 0$ with $np \to \lambda \in (0, \infty)$. Show that $P(X_n = k) \to P(X = k)$ as $n \to \infty$, for all $k \geq 0$. HINT: $(1 - \frac{c_n}{n})^n \to e^{-c}$ if $c_n \to c$.

(c) For n, p as in part (b), show that $P(X_n > k) \to P(X > k)$ as $n \to \infty$, for all $k \geq 0$.