Numerical Analysis Preliminary Examination
Fall 2019

Problem 1.

(a) Prove that the product of two lower triangular matrices is lower triangular.
(b) Prove that the inverse of a nonsingular lower triangular matrix is lower triangular.
(c) Prove that if A is a nonsingular matrix that can be put in row echelon form using elementary row operations but without interchanging any rows, then it can be factored as $A = LU$ where L is lower triangular and U is upper triangular with all ones down the diagonal. Such a factorization is called an LU decomposition of the matrix.
(d) Prove if A is positive definite that it has an LU decomposition.

Problem 2. Suppose $A \in \mathbb{C}^{m\times m}$ is nonsingular. Consider the iterative scheme

$$X_{n+1} = X_n + c(AX_n - I)$$

to calculate A^{-1}, where $X_0 \in \mathbb{C}^{m\times m}$ and $c \in \mathbb{C}$, $c \neq 0$.

(a) Find a necessary and sufficient condition on A and c under which the scheme converges to A^{-1} for all initial matrices X_0.
(b) Suppose the eigenvalues of A are all real with

$$1 = \lambda_r \leq \lambda_{r-1} \leq \ldots \leq \lambda_1 = 5.$$

What is the optimal value of $c \in \mathbb{C}$ that achieves the maximum rate of convergence?
(c) Suppose now that the eigenvalues of A are all real with

$$-1 = \lambda_r \leq \lambda_{r-1} \leq \ldots \leq \lambda_1 = 5.$$

Show there is no value of c that will make the scheme converge for all initial matrices X_0. But consider the modified scheme

$$X_{n+1} = X_n + C(AX_n - I)$$

where C is now a matrix instead of a number. Find a matrix C such that this scheme converges to A^{-1} for all initial matrices X_0.

Problem 3. Let $\{\varphi_i\}_{i=1}^m$ be m linearly independent vectors in $\mathbb{R}^n$ and set

$$\Phi = [\varphi_1|\varphi_2|...|\varphi_m] \in \mathbb{R}^{n\times m}.$$

(a) Prove that $\Phi^T\Phi$ is nonsingular.
(b) Let $||\cdot||$ denote the Euclidean norm on $\mathbb{R}^n$ and define the map $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P_\Phi x = \varphi, \quad \text{where} \quad \varphi = \arg \min_{\psi\in \text{span}\{\varphi_i\}_{i=1}^m} ||x - \psi||.$$

Show that P_Φ is a linear map.
(c) A linear transformation P on a Hilbert space is said to be an orthogonal projection if
  1) P is self adjoint (i.e. $P^* = P$) and
  2) P is idempotent (i.e. $P^2 = P$).

Show that P_Φ is an orthogonal projection.
(d) Find a set of vectors in $\mathbb{R}^n$ whose span is equal to the orthogonal complement of the subspace spanned by the $\{\varphi_i\}_{i=1}^m$.

Problem 4. The characteristic polynomial of a matrix $A \in \mathbb{C}^{5\times 5}$ has the form

$$p(\lambda) = (\lambda - 1)^3(\lambda - 2)^2.$$

(a) Find all possible Jordan canonical matrices J for A such that $J_{k,k} \geq J_{k+1,k+1}$, $k = 1 \ldots, n-1$.

(b) For each of the matrices J find the algebraic and geometric multiplicity of its eigenvalues.

(c) For each of the matrices J find the minimal annihilating polynomial p_0 such that $p_0(A) = 0$.