DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2018

Each problem is worth 20 points. There are 4 problems.

1. Consider the system of ODEs,

$$\begin{aligned} x' &= \mu x - y - (x^2+y^2)x, \\ y' &= x + \mu y - (x^2+y^2)y, \end{aligned} \tag{1}$$

where $\mu \in (-1, 1)$.

   (a) Show that trajectories never leave the box $[-A, A] \times [-B, B]$ for suitably chosen A and B.

   (b) Show that for $\mu > 0$ there exists a non-trivial closed trajectory Γ in the plane.

   (c) Describe the bifurcation that takes place as μ increases through $\mu = 0$.

   (d) Compute the approximate period of the closed cycle for $\mu = 0.001$.

   (e) Show that Γ in part (b) is locally *exponentially* attracting, i.e. nearby trajectories decay exponentially toward Γ.

2. Consider the system

$$\begin{aligned} x'(t) &= -y(t)z(t) \\ y'(t) &= x(t)z(t) \\ 3z'(t) &= -x(t)y(t). \end{aligned}$$

   (a) Show that the quantity $x^2(t)+4y^2(t)+9z^2(t)$ is conserved, i.e constant along orbits.

   (b) Find all stationary points and determine their type and stability.

3. For the system

$$\begin{aligned} x' &= 3(\sin^2 t)x - 3y \\ y' &= -(\sin^4 t)x - y, \end{aligned}$$

   (a) Show there is at least one solution $\psi(t) = (x(t), y(t))$ that satisfies $|\psi(t)| \to \infty$ as $t \to \infty$,

   (b) Show there can be no solution $\phi(t) = (x(t), y(t))$ that satisfies $x(t_0) = x'(t_0) = 0$ for some $t_0 \in \mathbb{R}$ except the identically zero solution.

4. For the system of

$$\begin{aligned} x' &= 2x + 5y + 5x^2 - 4y^2 \\ y' &= 3x + y - 6x^2 + 5y^2, \end{aligned}$$

   (a) Show there is at least one non-identically zero solution $\psi(t) = (x(t), y(t))$ that satisfies $|\psi(t)| \to 0$.

   (b) Is the origin asymptotically stable?