PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2018

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $g_j \in C(\partial\Omega)$ converge uniformly to $g \in C(\partial\Omega)$ (recall that this means that $\lim_{j\to\infty} \sup_{x\in\partial\Omega} |g_j(x) - g(x)|$). Let $u_j \in C^2(\Omega) \cap C(\overline{\Omega})$ be the solutions of

$$\begin{aligned} \Delta u_j &= 0 \text{ in } \Omega, \\ u_j &= g_j \text{ on } \partial\Omega. \end{aligned}$$

Show that u_j converges uniformly to a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and that u solves

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

2. Let $u_0 \in C^2(B_1(0)), u_1 \in C^1(B_1(0)), f \in C((0,T) \times B^1(0))$. Show that the problem

$$\begin{aligned} \partial_{tt} u - \Delta u + u &= f \text{ in } (0,T) \times B^1(0), \\ u(0,x) &= u_0(x), \\ \partial_t u(0,x) &= u_1(x), \\ u(t,x) &= 0 \text{ on } (0,T) \times \partial B_1(0), \end{aligned}$$

has at most one solution $u \in C^2([0,T] \times \overline{B_1(0)})$.

3. Consider the equation

$$\partial_t u + u \partial_x u = 0$$

on $(0,T) \times \mathbb{R}$. Show that a classical solution with initial data $u(0,x) = \frac{\pi}{2} - \arctan(x)$ can exist at most for a finite time.