1. Let $\{\mathcal{P}_\theta,\ \theta \in \Theta\}$ be a family of probability distributions, and $X \sim \mathcal{P}_\theta$ for some $\theta \in \Theta$. Let $\mathcal{E}$ be the set of all unbiased estimators of 0 with finite variances, and let T be an unbiased estimator of θ with $\text{Var}(T) < \infty$.

   (a) Assume that T is the uniformly minimum-variance unbiased estimator (UMVUE) of θ. Show that $\mathbb{E}_\theta[T(X)U(X)] = 0$ for any $U \in \mathcal{E}$ and $\theta \in \Theta$.

   (b) Assume that $\mathbb{E}_\theta[T(X)U(X)] = 0$ for any $U \in \mathcal{E}$ and $\theta \in \Theta$. Show that T is the UMVUE.

   (c) Assume now that the family of distributions is parametrized by k parameters $\theta_1, \ldots, \theta_k$, and that T_j is the UMVUE of θ_j, $1 \le j \le k$. Use (a) and (b) to show that $\sum_{j=1}^k \alpha_j T_j$ is the UMVUE of $\sum_{j=1}^k \alpha_j \theta_j$ for any $\alpha_1, \ldots, \alpha_k$.

2. Let $X_1, \ldots, X_n$ be independent with distribution equal to that of X, a random variable with mean $\mu = E[X]$, variance σ^2, and finite sixth moment $E[X^6] < \infty$. Define the skewness parameter

$$\tau = \frac{E[X-\mu]^3}{\sigma^3}.$$

   Assume μ is known, and consider the estimation of τ by

$$\widehat{\tau} = \frac{\widehat{m_3}}{\widehat{\sigma^3}} \quad \text{where} \quad \widehat{m_3} = \frac{1}{n}\sum_{i=1}^n (X_i-\mu)^3 \quad \text{and} \quad \widehat{\sigma^3} = \left(\frac{1}{n}\sum_{i=1}^n (X_i-\mu)^2\right)^{3/2}.$$

   (a) Show that the parameter τ is invariant with respect to translation and positive scaling of X.

   (b) Determine, and justify, whether or not $\widehat{\tau}$ is UMVU when the X distribution is normal.

   (c) Assuming the X distribution is symmetric about its mean, find the mean and variance of $(X_i-\mu)^2$ and $(X_i-\mu)^3$, and their covariance. Apply the multivariate central limit theorem to yield the limiting distribution of the properly scaled and centered (bivariate) sum

$$S_n = \frac{1}{n}\sum_{i=1}^n ((X_i-\mu)^2, (X_i-\mu)^3).$$

   (d) Recall that when a sequence of vectors $\mathbf{Y}_n$ in $\mathbb{R}^d$ converges in distribution to $\mathbf{Y}_0$, that is,

$$\sqrt{n}(\mathbf{Y}_n - E\mathbf{Y}_n) \to_d \mathbf{Y}_0,$$

   and $g : \mathbb{R}^d \to \mathbb{R}^r$ is a nice function, the multivariate delta method yields that

$$\sqrt{n}(g(\mathbf{Y}_n) - g(E\mathbf{Y}_n)) \to_d \dot{g}^T(E(\mathbf{Y}_0))\mathbf{Y}_0,$$

   where $g = (g_1, \ldots, g_r)$ and $\dot{g}$ is the matrix whose columns are the gradients of $g_1, \ldots, g_r$. That is, $\dot{g} = (\nabla g_1, \ldots, \nabla g_r)$. Use this result to derive a properly scaled limiting distribution for $\widehat{\tau}$.