Geometry and Topology Graduate Exam
Fall 2018

Solve all **6** problems. Partial credit will be given to partial solutions.

Problem 1. Let $p : \tilde{X} \to X$ denote the universal cover of the space X, and let $Y \subset X$ be a path-connected subspace. Suppose that the preimage $p^{-1}(Y)$ is path connected. Show that, for an arbitrary base point $y_0 \in Y$, the homomorphism $i_* : \pi_1(Y; y_0) \to \pi_1(X; y_0)$ induced by the inclusion map $i\colon Y \to X$ is surjective.

Problem 2. Consider the Klein bottle

$$K = S^1 \times [0,1]/\sim$$

where, if S^1 is the unit circle in the complex plane $\mathbb{C}$ and if $\bar{z}$ denotes the complex conjugate of $z \in S^1$, the equivalence relation $\sim$ identifies each $(z,1) \in S^1 \times \{1\}$ to $(\bar{z}, 0) \in S^1 \times \{0\}$. Compute all homology groups $H_n(K; \mathbb{Z}_4)$ with coefficients in the cyclic group $\mathbb{Z}_4$ of order 4.

Problem 3. Consider the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0_n & \mathrm{I}_n \\ -\mathrm{I}_n & 0_n \end{pmatrix}$$

where I_n denotes the identity matrix of order n, and where 0_n represents the square matrix of order n whose entries are all equal to 0. In the vector space $\mathrm{M}_{2n}(\mathbb{R}) \cong \mathbb{R}^{4n^2}$ of $2n \times 2n$ matrices, set

$$\Sigma_n = \{A \in \mathrm{M}_{2n}(\mathbb{R}); AJA^{\mathrm{t}} = J\} = f^{-1}(J)$$

for the map $f\colon \mathrm{M}_{2n}(\mathbb{R}) \to \mathrm{M}_{2n}(\mathbb{R})$ defined by $f(A) = AJA^{\mathrm{t}}$.

- **a.** Let $T_{\mathrm{I}_{2n}} f\colon \mathrm{M}_{2n}(\mathbb{R}) \to \mathrm{M}_{2n}(\mathbb{R})$ denote the tangent map (= differential map) of f at the identity matrix $\mathrm{I}_{2n} \in \mathrm{M}_{2n}(\mathbb{R})$ where, since $\mathrm{M}_{2n}(\mathbb{R})$ is a vector space, we use the canonical identification between the tangent space $T_{\mathrm{I}_{2n}}\mathrm{M}_{2n}(\mathbb{R})$ and $\mathrm{M}_{2n}(\mathbb{R})$. Determine the dimension of the image of $T_{\mathrm{I}_{2n}} f$.
- **b.** Show that there is a neighborhood U of I_{2n} in $\mathrm{M}_{2n}(\mathbb{R})$ such that $\Sigma_n \cap U$ is a submanifold of $\mathrm{M}_{2n}(\mathbb{R}) \cong \mathbb{R}^{4n^2}$. What is its dimension?

Problem 4. Let M be an m–dimensional submanifold of $\mathbb{R}^n$. Show that, if $n > 2m+1$, there exists a hyperplane $H \subset \mathbb{R}^n$ such that the restriction of the orthogonal projection $\pi_H\colon \mathbb{R}^n \to H$ to M is injective. Possible hint: consider the map which associates the vector $\overrightarrow{PQ}/\|\overrightarrow{PQ}\|$ to each pair $(P,Q) \in M \times M$ with $P \neq Q$.

Problem 5. Let M be a compact, oriented, smooth manifold with boundary ∂M. Prove that there does not exist a map $F : M \to \partial M$ such that $F_{|\partial M}\colon \partial M \to \partial M$ is the identity map.

Possible hint: consider the inclusion map $i\colon \partial M \to M$, and use Stokes's theorem.

Problem 6. Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit sphere, and let $\omega \in \Omega^n(S^n)$ be the differential n-form given by the restriction of $x_{n+1} dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{R}^{n+1})$ to S^n. Show that the class of ω in the de Rham cohomology $H^n_{\mathrm{dR}}(S^n)$ is nontrivial, namely that ω is closed but not exact.