REAL ANALYSIS

Fall 2018

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f : [0,1] \to \mathbf{R}$ be an absolutely continuous function. Let

$$g(x) = \int_0^1 f(xt)\,dt, x \in [0,1].$$

Show that g is an absolutely continuous function.

2. Let f be Lebesgue measurable on $[0,1]$ and assume $f(x) > 0$ for almost every x. Let E_k for $k = 1, 2, \ldots$ be a sequence of measurable sets on $[0,1]$ so that $\int_{E_k} f(x)\,dx \to 0$ as $k \to \infty$. Show that $m(E_k) \to 0$ as $k \to \infty$.

3. Let $f \in L^1(\mathbf{R})$, and let

$$\begin{aligned} S_n(x) &= \frac{1}{n}\sum_{j=0}^{n-1} f\left(x + \frac{j}{n}\right), x \in \mathbf{R}, \\ S(x) &= \int_x^{x+1} f(y)\,dy, x \in \mathbf{R}. \end{aligned}$$

Show that $f_n \to f$ in $L^1(\mathbf{R})$.

4. Assume that f_n is a sequence of integrable functions on $\mathbf{R}$ such that

$$\lim_{n\to\infty} \int f_n(x)\,g(x)\,dx = g(0)$$

for all g continuous with compact support.

Prove that f_n is not a Cauchy sequence in $L^1(\mathbf{R})$.