COMPLEX ANALYSIS

Fall 2018

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $a > 0$. Compute

$$\int_0^{\pi} \frac{d\theta}{a^2 + \sin^2 \theta}.$$

2. Find the number of solutions of the equation $z - 2 - e^{-z} = 0$ in $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

3. Let $\Omega \neq \mathbb{C}$ be simply connected and let for any $c \in \Omega$, the mapping $\phi_c : \Omega \to \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be conformal so that $\phi_c(c) = 0$. Let $g_c(z) = \log |\phi_c(z)|$, $z \in \Omega \setminus \{c\}$.

Show that $g_a(b) = g_b(a)$ for any distinct $a, b \in \Omega$.

4. Let $a \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and

$$f_a(z) = \frac{a - z}{1 - \bar{a}z}, z \in \bar{\mathbb{D}}.$$

Show that f_a is a holomorphic bijective mapping of $\mathbb{D}$ onto $\mathbb{D}$ which is its own inverse.