Algebra Qualifying Exam - Fall 2018

1. Let $\mathbb{F}_p$ be a finite field with p elements, and consider the group $GL_n(\mathbb{F}_p)$. Write down the order of $GL_n(\mathbb{F}_p)$ and a Sylow p-subgroup.

2. Prove that there are no simple groups of order 600.

3. Prove that $\mathbb{Z}[\sqrt{10}]$ is integrally closed in its field of fractions, but not a UFD.

4. If F is a field and E/F is an extension, then an element $a \in E$ will be called abelian if $Gal(F[a]/F)$ is an abelian group. Show that the set of abelian elements of E is a subfield of E containing F.

5. Let K be the splitting field of $x^4 - 2 \in \mathbb{Q}[x]$. Prove that $Gal(K/\mathbb{Q})$ is D_8 the dihedral group of order 8 (i.e., the group of isometries of the square). Find all subfields of K that have degree 2 over $\mathbb{Q}$.

6. Let F be a field, and suppose A is a finite-dimensional F-algebra. Write $[A, A]$ for the F-subspace of A spanned by elements of the form $ab - ba$ with $a, b \in A$. Show that $[A, A] \neq A$ in the following two cases:

   (a) when A is a matrix algebra over F;

   (b) when A is a central division algebra over F.

   (Recall that a division algebra over F is called central if its center is isomorphic with F.)

7. If $\varphi : A \to B$ is a surjective homomorphism of rings, show that the image of the Jacobson radical of A under φ is contained in the Jacobson radical of B.