MATH 507a **QUALIFYING EXAM** **August 2018**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

1. (a) Let $X_1, X_2, \ldots$ be a sequence of random variables defined on the same probability space. Show that there exists a sequence $a_1, a_2, \ldots$ of real numbers such that $X_n/a_n \to 0$ as $n \to \infty$ almost surely as $n \to \infty$.

(b) Suppose $Y_1, Y_2, \ldots$ are independent exponential random variables with the same mean $\lambda > 0$. Find

$$\liminf_{n\to\infty} \frac{Y_n}{\log n} \text{ and } \limsup_{n\to\infty} \frac{Y_n}{\log n}.$$

2. (a) Define the characteristic function $\phi_X(t)$ of a random variable X.

(b) Show that if X_n converges to X in distribution then $\phi_{X_n}(t)$ converges to $\phi_X(t)$ for all t.

(c) Suppose conversely that the characteristic functions $\phi_{X_n}(t)$ converge as $n \to \infty$ for all t. Give an example showing that this does not imply that the X_n converge in distribution. Give also an extra condition under which the X_n do converge in distribution.

3. Suppose that $X_0, X_1, X_2, \ldots$ are independent and identically distributed random variables with the uniform distribution on $[0, 1]$. Consider the random variable

$$N = \inf\{n > 0 : X_n > X_0\}.$$

(a) Find the probability mass function for N.

(b) Find $\mathbb{E}\, N$.

(c) Find $\mathbb{E}\left(\lfloor\sqrt{N}\rfloor\right)$, where $\lfloor\cdot\rfloor$ denotes the floor function, that is, $\lfloor x\rfloor =$ the greatest integer not exceeding x.

(d) Find the joint probability $\mathbb{P}(X_0 \le t, N = n)$ for $0 < t < 1, n = 1, 2, \ldots$.