MATH 505a **QUALIFYING EXAM** **August 2018**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

1. Let X be an exponentially distributed random variable with $\mathbb{P}(X > t) = e^{-rt}$ for $t > 0$. Write X as the sum of its integer and fractional parts: $X = Y + Z$ with $Y = \lfloor X \rfloor \in \mathbb{Z}$ and $Z \in [0, 1)$.

(a) Find $\mathbb{E}\, X$

(b) Find $\mathbb{P}(Y = n)$, $n = 0, 1, 2, \ldots$.

(c) Find $\mathbb{E}\, Y$ and $\mathbb{E}\, Z$.

(d) Show that Y and Z are independent.

2. Let f and g be bounded nondecreasing functions on $\mathbb{R}$, and let X, Y be independent and identically distributed random variables.

(a) Show that

$$\mathbb{E}\left[\Big(f(X) - f(Y)\Big)\Big(g(X) - g(Y)\Big)\right] \geq 0.$$

(b) Show that $f(X)$ and $g(X)$ are positively correlated, that is,

$$\mathbb{E}\Big[f(X)g(X)\Big] \geq \mathbb{E}\left[f(X)\right] \cdot \mathbb{E}\left[g(X)\right].$$

3. Suppose that X and Y have joint density $f(x, y)$ given by $f(x, y) = ce^{-x}$ for $x > 0$ and $-x < y < x$ and $f(x, y) = 0$ otherwise.

(a) Show that $c = 1/2$.

(b) Find the marginal densities of X and Y, and the conditional density of Y given X.

(c) Find $\mathbb{P}(X > 2Y)$.