Numerical Analysis Screening Examination
Fall 2018-2019

1. (Numerical methods for finding eigenvalues) Let A be a square $n \times n$ matrix with eigenvalue-eigenvector pairs $\{(\lambda_i, u_i)\}_{i=1}^n$ satisfying $\{u_i\}_{i=1}^n$ linearly independent and $|\lambda_1| > |\lambda_2| \geq \cdots . \geq |\lambda_n|$. Let $x_0 \in C^n$ be given with $x_0 = \sum_{i=0}^n \alpha_i u_i$ and $\alpha_1 \neq 0$. For $k = 1,2, \ldots$, set $\beta_k = x_{k-1}^T x_k / x_{k-1}^T x_{k-1}$, where $x_k = Ax_{k-1}$.
   a. Show that $\beta_k = \lambda_1 \left(1 + O\left(|\lambda_2/\lambda_1|^k\right)\right)$, as $k \to \infty$.
      (Recall that if $h > 0$, $r_k = O\left(h^k\right)$, as $k \to \infty$, if and only if there exists a positive integer k_0 and a positive constant M such that $|r_k| \leq Mh^k$, for all $k > k_0$, or, equivalently if and only if $\frac{|r_k|}{h^k}$ is bounded for all positive integers k.)
   b. Show that if the matrix A is symmetric, then $\beta_k = \lambda_1 \left(1 + O\left(|\lambda_2/\lambda_1|^{2k}\right)\right)$, as $k \to \infty$.
   c. Let α be a given complex number. Show how an iteration like the one given above can be used to find the eigenvalue of A that is closest to α.

2. (Iterative methods for linear systems) A square matrix A is said to be *power bounded* if all the entries in A^m remain bounded as $m \to \infty$.
   a. Show that if $\|A\| < 1$, where $\| \quad \|$ is some induced matrix norm, then A is power bounded.
   b. Establish necessary and sufficient conditions on the spectrum of a diagonalizable matrix A to be power bounded.
   c. For λ a complex number and k a nonnegative integer, let $J_k(\lambda)$ denote the $k \times k$ matrix with λ's on the diagonal and 1's on the first super diagonal, and show that
   $$J_k(\lambda)^m = \sum_{j=0}^{k-1} \binom{m}{m-j} \lambda^{m-j} J_k(0)^j$$
   d. Find necessary and sufficient conditions for an arbitrary square matrix A to be power bounded.

3. (Least squares) Consider the following least square minimization problem
   $$\min_{x \in \mathbb{R}^4} \|Ax - b\|_2^2$$
   where
   $$A = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.$$
   a. Explain why the problem has a solution.
   b. Determine whether or not $x_0 = (1,1,0,0)$ is a solution.
   c. Determine whether or not x_0 is the minimum norm solution to this problem.
   d. Find the minimum norm solution.

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4. (Direct methods for linear systems)

a) Consider a block matrix

$$K = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

where E, F, G, and H are all square $n \times n$ matrices. Show that, in general,

$$\det(K) \neq \det(E)\det(H) - \det(F)\det(G)$$

but if either F or G (or both) is the zero matrix (so K is either block-lower or block-upper triangular) then

$$\det(K) = \det(E)\det(H).$$

b) Suppose that A is a non-singular $n \times n$ matrix and B is any $n \times n$ matrix such that the $2n \times 2n$ matrix

$$C = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is also non-singular. By considering the matrix

$$\begin{bmatrix} I & 0 \\ -BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

or otherwise, show that

$$\det(C) = [\det(A)]^2 \det(I - A^{-1}BA^{-1}B).$$

c) Now suppose that A and B are any $n \times n$ matrices such that the $2n \times 2n$ matrix C given in Part b is nonsingular. Use Part a to show that both of the matrices $A + B$ and $A - B$ must be non-singular.

d) Consider the system of equations $Cx = b$ where the matrix $C$is as given in Part c above. Let $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ where b_1, b_2 are in $\mathbb{R}^n$ and let y_1 and y_2 be the unique solutions to

$$(A + B)y_1 = b_1 + b_2 \quad \text{and} \quad (A - B)y_2 = b_1 - b_2$$

guaranteed to exist by Part c above. Show how to obtain the solution of $Cx = b$ from y_1 and y_2. What is the numerical advantage of finding the solution of $Cx = b$ in this way rather than finding it directly?