PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2017

Choose THREE out of four problems: only three will be graded. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider the Burgers' equation

$$u_t + uu_x = 0, \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = g(x), \qquad x \in \mathbb{R}.$$

where $g(x)$ is a given piecewise continuous function.

(a) (5pts) Show that the characteristics are straight lines.

(b) (2.5pts) Give an example of $g(x)$ so that the characteristics do not cover the entire (x,t) space.

(c) (2.5pts) Give an example of $g(x)$ so that the characteristics intersect.

2. Let $U \subset \mathbb{R}^n$ be an open set.

(a) (5pts) Let $u \in C^2(U)$. Show that for any Ball $\overline{B}(x_0, r) \subset U$ it holds

$$\frac{d}{dr} \fint_{\partial B(0,1)} u(x_0 + rz) d\mathcal{S}(z) = \frac{r}{n} \fint_{B(0,1)} (\Delta u)(x_0 + rz) dz.$$

Here we have used the notation $\fint_A f = \frac{1}{|A|} \int_A f$.

(b) (3pts) Let $u \in C^2(U)$ be such that for any ball $\overline{B}(x_0, r) \subset U$ it holds

$$u(x_0) = \fint_{\partial B(x_0,r)} u d\mathcal{S}.$$

Show that then $\Delta u = 0$ in U.

(c) (2pts) Does the implication of part (b) still hold if you just assume $u \in C(U)$? Briefly explain your answer.

3. Let U be the unit ball in $\mathbb{R}^n$.

(a) (5pts) For $u(x) = |x|^{-a}$ for $x \in U$, determine the values of $a,\ n,\ p$ for which u belongs to the Sobolev space $W^{1,p}(U)$.

(b) (5pts) Let $n \geq 2$. If $u(x) = \ln \ln(1 + \frac{1}{|x|})$ for $x \in U$, show that $u \in W^{1,n}(U)$ but not in $L^\infty(U)$.

4. (a) (5pts) Let $U \subset \mathbb{R}^n$ be a bounded open set, let $T > 0$ be fixed and define $U_T = U \times (0, T)$. Assume $u \in C_1^2(\overline{U_T})$ solves the following initial boundary value problem

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in} \quad U_T, \\ \frac{\partial u}{\partial \nu} + u &= h \quad \text{on} \quad \partial U \times (0, T), \\ u &= g \quad \text{on} \quad U \times \{t = 0\}, \end{aligned}$$

where f, g, and h are given smooth functions, and ν is the outward pointing unit normal field of ∂U. Prove that there exists at most one such solution.

(b) (5pts) Let $U \subset \mathbb{R}^2$ be a bounded open set and let $a > 0, b, c \in \mathbb{R}$ be given constants. Show that any solution $u \in C^2(\overline{U})$ of

$$\Delta u - au + b\partial_x u + c\partial_y u = 0 \quad \text{in} \quad U,$$

cannot attain a positive maximum or negative minimum inside U.