1. For known $\alpha > 0$ and unknown $\theta \in \mathbb{R}$, we observe the sequence $X_1, \ldots, X_n$ of independent variables with distribution function

$$F(x;\theta) = \begin{cases} 0 & x \le \theta \\ (x-\theta)^\alpha & x \in (\theta, \theta+1] \\ 1 & x > \theta + 1, \end{cases}$$

and consider the estimator of θ given by

$$\widehat{\theta} = \min\{X_1, \ldots, X_n\}.$$

   a. Determine the distribution function of $\widehat{\theta}$.

   b. Determine a non-trivial distribution for a random variable Y and an increasing sequence a_n of real numbers such that

$$a_n(\widehat{\theta} - \theta) \to_d Y.$$

   c. What rates of (distributional) convergence to θ are possible for the estimator $\widehat{\theta}$ as α ranges over $(0, \infty)$?

2. Let $X_1, \ldots, X_n$ be i.i.d. with the $N(\theta, \theta)$ distribution, for some $\theta > 0$.

   (a) Find the MLE of θ.

   (b) Show that the MLE is a consistent estimator of θ.

   (c) Assume that $n = 1$.

      i. Show that $T(X) = |X|$ is a sufficient statistic for θ.

      ii. Note that $\hat{\theta}(X) = X$ is an unbiased estimator of θ, but is not a function of $T(X)$. Hence, it can be improved; apply the Rao-Blackwell theorem to find another estimator θ^* that is a function of $T(X)$ and is unbiased for θ.