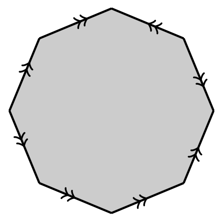
Geometry and Topology Graduate Exam
Fall 2017

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Problem 1. Let M be an oriented compact m–dimensional manifold, and let $f: M \to \mathbb{R}^m$ be a smooth map. Show that, for almost every $y \in \mathbb{R}^m$ (meaning, for y in the complement of a set of measure 0), the preimage $f^{-1}(y)$ consists of an *even* number of points.

Problem 2.
The sides of an octagon are glued using the pattern below. Determine the fundamental group of the associated quotient space.



Problem 3. Let $p: \widetilde{X} \to X$ be a covering map with X path connected and locally path connected, and with $\pi_1(X; x_0) \cong \mathbb{Z}/5$. Show that, if the fiber $p^{-1}(x_0)$ consists of 4 points, the covering is trivial.

Problem 4. Consider the following two-dimensional distribution on $\mathbb{R}^3$:

$$\mathcal{D} = \ker(2dx - e^y dz).$$

Is there a neighborhood U of $0 \in \mathbb{R}^3$, along with a coordinate system (w, s, t) on U, such that $\mathcal{D}|_U = \text{span}(\frac{\partial}{\partial w}, \frac{\partial}{\partial s})$? Justify your answer (with a proof).

Problem 5. Show that the subset

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1^2 + x_2^2 = x_3^2 + x_4^2\}$$

is *not* a differentiable submanifold of $\mathbb{R}^4$.

Problem 6. Let X be the subspace of $\mathbb{R}^3$ defined by

$$X = \left\{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 - 1)(x^2 + z^2 - (\tfrac{1}{2})^2) = 0, \right\}$$

so that X is the union of two cylinders of radius 1 along the z-axis and a cylinder of radius $\frac{1}{2}$ along the y-axis. Determine the homology groups $H_*(X)$.

Problem 7. Consider the 2-form on $\mathbb{R}^3 \backslash \{(0,0,0)\}$ given by

$$\sigma = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \left(x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy\right).$$

Show that σ is closed but not exact. (Possible hint: Integrate σ over the sphere S^2).