REAL ANALYSIS GRADUATE EXAM

Fall 2017

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $(X, \mathscr{A}, \mu)$ be a measure space and f, g, f_n, g_n measurable so that $f_n \to f$ and $g_n \to g$ in measure. Is it true that $f_n^3 + g_n \to f^3 + g$ in measure if
(i) $\mu(X) = 1$
(ii) $\mu(X) = \infty$.
In both cases prove the statement or provide a counter example.

2. Let $f \in \mathscr{L}^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} f(x+n)$$

converges absolutely for Lebesgue almost every $x \in \mathbb{R}$.

3. Assume that $E \subset \mathbb{R}$ is such that $m(E \cap (E+t)) = 0$ for all $t \neq 0$, where m is the Lebesgue measure on $\mathbb{R}$. Prove that $m(E) = 0$.

4. Let $(X, \mathscr{A}, \mu)$ be a measure space and f_n a sequence of non-negative measurable functions. Prove that if $\sup_n f_n$ is integrable then

$$\limsup_n \int_X f_n \, d\mu \leq \int_X \limsup_n f_n \, d\mu.$$

Also show that
(i) the inequality may be strict and
(ii) that the inequality may fail unless $\sup_n f_n \in \mathscr{L}^1$.