Algebra Qualifying Exam - Fall 2017

1. Assume S is a commutative integral domain, and $R \subset S$ is a subring. Assume S is finitely generated as an R-module, i.e., there exist elements $s_1, \ldots, s_n \in S$ such that $S = s_1R + s_2R + \cdots + s_nR$. Show that R is a field if and only if S is a field. Is the statement true if the assumption that S is an integral domain is dropped?

2. Suppose R is a commutative unital ring, $\mathfrak{p} \subset R$ is a prime ideal and M is a finitely generated R-module. Recall that the annihilator ideal $\mathrm{Ann}_R(M)$ consists of elements $r \in R$ such that $rm = 0$ for all $m \in M$. Show the localized module $M_{\mathfrak{p}}$ is *non-zero* if and only if $\mathrm{Ann}_R(M) \subset \mathfrak{p}$.

3. Let $f(x) = x^5 + 1$. Describe the splitting field K of $f(x)$ over $\mathbb{Q}$ and compute the Galois group $Gal(K/\mathbb{Q})$.

4. Let α be the real positive 16th root of 3 and consider the field $F = \mathbb{Q}(\alpha)$ generated by α over the field of rational numbers. Observe that there is a chain of intermediate fields

$$\mathbb{Q} \subset \mathbb{Q}(\alpha^8) \subset \mathbb{Q}(\alpha^4) \subset \mathbb{Q}(\alpha^2) \subset \mathbb{Q}(\alpha) = F.$$

   Compute the degrees of these intermediate field extensions and conclude they are all distinct. Show that every intermediate field K between $\mathbb{Q}$ and F is one of the above (hint: consider the constant term of the minimal polynomial of α over K)

5. A finite group is said to be *perfect* if it has no nontrivial abelian homomorphic image. Show that a perfect group has no non-trivial solvable homomorphic image. Next, suppose that $H \subset G$ is a normal subgroup with G/H perfect. If $\theta : G \to S$ is a homomorphism from G to a solvable group S and if $N = \ker \theta$, show that $G = NH$ and deduce that $\theta(H) = \theta(G)$.

6. Let A be a finite-dimensional $\mathbb{C}$-algebra. Given $a \in A$, write L_a for the left-multiplication operator, i.e., $\mathrm{L}_a(b) = ab$. Define a map $(-,-) : A \times A \to \mathbb{C}$ by means of the formula $(a, b) := \mathrm{Tr}\, \mathrm{L}_a \mathrm{L}_b$.
   (a) Show that $(-,-)$ is a symmetric bilinear form on A.
   (b) If one defines the radical $\mathrm{Rad}(-,-)$ as $\{a \in A | (a, b) = 0\ \forall b \in A\}$, then show that $\mathrm{Rad}(-,-)$ is a two-sided ideal in A.
   (c) Show that $\mathrm{Rad}(-,-)$ coincides with the Jacobson radical of A.

7. Suppose F is an algebraically closed field, V is a finite-dimensional F-vector space, and $A \in \mathrm{End}_F(V)$. Show that there exist polynomials $f, g \in F[x]$ such that i) $A = f(A) + g(A)$, ii) $f(A)$ is diagonalizable and $g(A)$ nilpotent, and iii) f and g both vanish at 0.